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Slowly varying control parameters, delayed bifurcations, and the stability of spikes in reaction–diffusion systems

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HIGHLIGHTS

- We present three examples of delayed bifurcations in partial differential equations.
- Bifurcations may be triggered by extrinsic tuning or intrinsic dynamics.
- Explicitly solvable nonlocal eigenvalue problems yield analytic results.
- Delay may play significant role in determining the eventual fate of the system.

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ABSTRACT

We present three examples of delayed bifurcations for spike solutions of reaction–diffusion systems. The delay effect results as the system passes slowly from a stable to an unstable regime, and was previously analyzed in the context of ODE's in Mandel and Erneux (1987). It was found that the instability would not be fully realized until the system had entered well into the unstable regime. The bifurcation is said to have been "delayed" relative to the threshold value computed directly from a linear stability analysis. In contrast to the study of Mandel and Erneux, we analyze the delay effect in systems of *partial* differential equations (PDE's). In particular, for spike solutions of singularly perturbed generalized Gierer–Meinhardt and Gray–Scott models, we analyze three examples of delay resulting from slow passage into regimes of oscillatory and competition instability. In the first example, for the Gierer–Meinhardt model on the infinite real line, we analyze the delay resulting from slowly tuning a control parameter through a Hopf bifurcation. In the second example, we consider a Hopf bifurcation of the Gierer–Meinhardt model on a finite one-dimensional domain. In this scenario, as opposed to the *extrinsic* tuning of a system parameter through a bifurcation value, we analyze the delay of a bifurcation triggered by slow *intrinsic* dynamics of the PDE system. In the third example, we consider competition instabilities triggered by the extrinsic tuning of a feed rate parameter. In all three cases, we find that the system must pass well into the unstable regime before the onset of instability is fully observed, indicating delay. We also find that delay has an important effect on the eventual dynamics of the system in the unstable regime. We give analytic predictions for the magnitude of the delays as obtained through the analysis of certain explicitly solvable nonlocal eigenvalue problems (NLEP's). The theory is confirmed by numerical solutions of the full PDE systems.

1. Introduction

The stability and bifurcation analysis of differential equations is one of the cornerstones of applied mathematics. In many applications, the bifurcation parameter is slowly changing, either extrinsically (e.g. parameter is experimentally controlled) or intrinsically (e.g. the bifurcation parameter is actually a slowly-changing variable). In these situations, the system can exhibit a significant delay in bifurcation: the instability is observed only as the parameter is increased well past the threshold predicted by the linear bifurcation theory, if at all. Often referred to as the slow passage through a bifurcation, and first analyzed in [1,2], there is a growing literature on this subject (see [3] for a recent overview of the subject and references therein). Some applications of delayed bifurcations include problems in laser dynamics [2], delayed chemical

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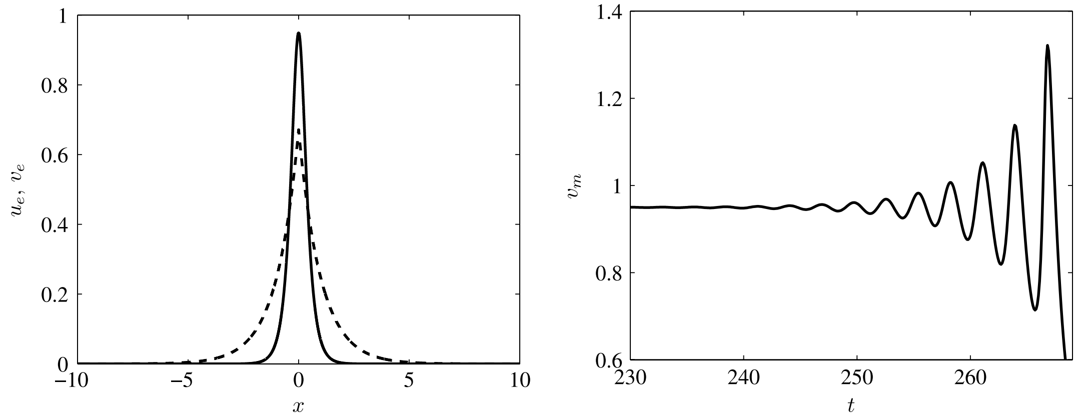

(a) $u_e(x)$ and $v_e(x)$.

(b) Amplitude oscillations in time.

Fig. 1. (a) The asymptotic equilibrium solution of v (solid) and u (dashed) for (2.1) with $\varepsilon = 0.3$. The width of the spike in v_e is proportional to ε, while u_e is independent of ε. Both v_e and u_e are independent of τ. (b) Typical example of amplitude oscillations in time when $\tau > \tau_H \approx 2.114$. The quantity plotted on the vertical axis is the height v_m of the spike in the left figure.

reactions [4], bursting oscillations in neurons [5], and noise-induced delay of the pupil light reflex [6], and early-warning signals [7].

Delayed bifurcation phenomena is relatively well understood in the context of ODE's. However much less is known in the context of PDE's. Motivated by the FitzHugh Nagumo (FHN) ODE example of [2], a particular case of the spatially extended FHN model was considered in [8]. The theoretical work showed that delayed Hopf bifurcations are possible also when the external stimulus, restricted to the form $I(x,t) = \phi(x)i(t)$, is spatially non-homogeneous. In [9], delayed Hopf bifurcations in an FHN PDE model were analyzed from a mainly numerical perspective by approximating the PDE as a large system of ODE's. In [10], predictions for delay in a general singularly perturbed scalar reaction–diffusion equation were obtained, where the stationary state in consideration was the spatially homogeneous zero state. In contrast, the main goal of this paper is to study in detail three representative examples of delayed bifurcations of spatially inhomogeneous states in PDE's, where explicit asymptotic results are obtainable which can be verified by numerical computations.

In order to present our examples both analytically and numerically, we focus on slight variants of the Gierer–Meinhardt (GM) and the Gray–Scott (GS) reaction–diffusion (RD) models. The GM system serves as a model for hydra head formation [11], sea shell pattern formation, and other biological processes [12,13], and falls within the framework of Turing's mathematical theory of morphogenesis [14]. The relevance of the GS model to laboratory experiments of the ferrocyanide–iodate–sulfite reaction are detailed in [15,16].

While we focus on these two models, the phenomena that we present in this paper are expected to be representative of a larger class of RD systems. The specific systems that we consider are

$$\text{GM model: } v_t = \varepsilon^2 v_{xx} - v + \frac{v^p}{u^q}, \quad \tau u_t = D u_{xx} - u + \frac{1}{\varepsilon}\frac{v^r}{u^s}, \tag{1.1}$$

and

$$\text{GS model: } v_t = \varepsilon^2 v_{xx} - v + A u^q v^p, \quad \tau u_t = D u_{xx} + 1 - u + \frac{1}{\varepsilon} u^s v^r, \tag{1.2}$$

for certain choices of the exponents p, q, r, and s (see below). In the singular limit $\varepsilon \to 0$, both of these models have equilibria that consist of spike solutions, characterized by an $\mathcal{O}(\varepsilon)$ width localization of v as ε^2 becomes asymptotically small. The component u varies over a comparatively long spatial scale and is independent of ε. In all three of our examples, we consider spike solutions that are qualitatively similar to that shown in Fig. 1(a).

To illustrate the main complications when generalizing delayed bifurcations to PDE's, let us first review the following prototypical ODE example [2]: $\frac{du}{dt} = (-1 + \varepsilon t)\, u$, $u(0) = u_0$ where $\varepsilon > 0$ is a small parameter. Here, the equilibrium state is $u = 0$ and can be thought of having an "eigenvalue" $\lambda(\varepsilon t) = -1 + \varepsilon t$ which grows slowly in time, and becomes positive as t is increased past $t = 1/\varepsilon$, at which point the steady state becomes "unstable". On the other hand, the exact solution is given by

$$u(t) = u_0 \exp\left\{\frac{(\varepsilon t - 1)^2 - 1}{2\varepsilon}\right\}, \tag{1.3}$$

which starts to grow rapidly only when the term inside the curly brackets becomes positive, that is at $t = 2/\varepsilon$, well after the bifurcation threshold of $t = 1/\varepsilon$. The difference between $2/\varepsilon$ and $1/\varepsilon$ is precisely the delay in bifurcation, and is inversely proportional to the growth rate ε.

We make two remarks on (1.3). First, note that $u(t)$ remains of order $\mathcal{O}(e^{-1/\varepsilon})$ when $0 < t < 2/\varepsilon$, and returns to its original amplitude u_0 only when $t = 2/\varepsilon$. We therefore say that the bifurcation is fully realized, or has fully set in, when $t = 2/\varepsilon$, and in turn define the delay as the difference between this time and the time $t = 1/\varepsilon$ at which the system just enters the unstable state. This definition is motivated by the following imagined scenario. Suppose that an experimenter in a laboratory setting attempts to find the bifurcation point of a physical system obeying the above ODE by perturbing the measured quantity u and slowly increasing a control parameter corresponding to λ. In order to maintain the system close to steady state, the initial perturbation is kept as small as possible, only slightly exceeding the sensitivity threshold of the measuring device used to monitor u. According to (1.3), the quantity of interest u is then expected to dip below the sensitivity threshold and remain below it until u returns to approximately its original value u_0. This occurs when $t = 2/\varepsilon$, well after the theoretical bifurcation that occurred at $t = 1/\varepsilon$. The difference between these two times is then of great importance, as it is directly related to the amount by which the experimenter will have missed the bifurcation.

Second, the solution (1.3) takes the form $u = ce^{\psi(\varepsilon t)/\varepsilon}$. The form of the time dependence motivates a WKB-type ansatz for slow passage through bifurcation points in more general systems. Suppose