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Nonautonomous analysis of steady Korteweg–de Vries waves under nonlocalised forcing

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HIGHLIGHTS

- Obtains analytical approximations for near-uniform and near-solitary wave profiles.
- Genuinely nonautonomous approach associates with hyperbolic and homoclinic solutions.
- Computed using normal and tangential deformations of stable and unstable manifolds.
- Small forcing need not have compact support, decay at infinity, or differentiability.
- Provides a tool for finding the number of solitary waves for a given small forcing.

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ABSTRACT

Recently developed nonautonomous dynamical systems theory is applied to quantify the effect of bottom topography variation on steady surface waves governed by the Korteweg–de Vries (KdV) equation. Arbitrary (but small) nonlocalised bottom topographies are amenable to this method. Two classes of free surface solutions – hyperbolic and homoclinic solutions of the associated augmented dynamical system – are characterised. The first of these corresponds to near-uniform free-surface flows for which explicit formulæ are developed for a range of topographies. The second corresponds to solitary waves on the free surface, and a method for determining their number is developed. Formulæ for the shape of these solitary waves are also obtained. Theoretical free-surface profiles are verified using numerical KdV solutions, and excellent agreement is obtained.

1. Introduction

The Korteweg–de Vries (KdV) equation is an archetypical evolution equation representing the balance of dispersion and weak nonlinearity in physical systems that generate waves. The physical motivation behind the original derivation of Korteweg and de Vries [1] was to describe long waves propagating in a rectangular channel, but there are also applications in ion acoustic waves in plasma, acoustic waves on a crystal lattice, and coupled oscillators [2–6]. The richness of the behaviour of solutions is such that there continues to be ongoing numerical and analytical studies to the KdV equation [7–13, cf.].

The form we examine in this article is the steady forced KdV equation represented in the dimensionless form [4,5,14–17,11,18]

$$\eta_{xxx} + 9\eta\eta_x - 6\,(F-1)\,\eta_x = -3p_x. \tag{1}$$

The equation approximates the elevation $\eta(x)$ of the interface between the water and air (free-surface) in a two-dimensional (x, y) gravity affected channel flow. The flow can be characterised with the dimensionless Froude number $F = U/\sqrt{gH}$, where U and H are the uniform flow speed and depth in the far field, respectively, and g is the acceleration due to gravity. We focus here specifically on the role of the forcing term $p(x)$, which represents either bottom topography, or an external surface pressure on the free-surface. The existence of a solitary wave solution is well-established for $p \equiv 0$ [1,19]; in this article, we find analytical approximations for solitary waves and allied solutions when $p \neq 0$, but is small.

When there is no forcing (i.e., with $p \equiv 0$), (1) is autonomous and is well-understood in terms of the autonomous (η, η_x) phase plane (Fig. 1(b)). An analysis for determining free-surface solutions can be performed in the *same* autonomous phase plane when p takes on very specific forms. In the case of Dirac delta forcing (corresponding to a localised forcing or bump in bottom topography), solutions can be rationalised as jumping from one

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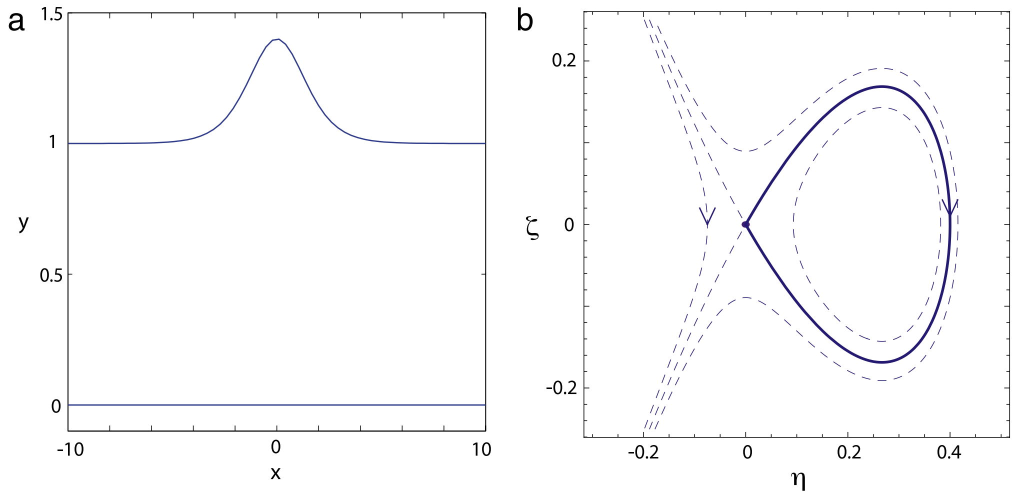

Fig. 1. Unforced solitary wave, with $y = \eta + 1$, $p(x) \equiv 0$ and $F = 1.2$. (a) Free-surface profile. (b) Autonomous phase space $\zeta = \eta_x = \eta'$ versus η.

solution trajectory to another in the autonomous phase space [5,16,17,20]. A jump also results from the presence of an inclined plane on the surface [21]. If p represents a vertical step at the bottom, solutions can be formed through the intersection of solution trajectories belonging to two different autonomous phase spaces [22,23]. Combinations of these three types of forcing (bump, plate and step) have also been studied in a similar way for hybrid flows with multiple disturbances [23–26], with an obvious limitation to a more general type of nonlocalised forcing.

A specific class of nonlocalised p has received some attention in the literature: sinusoidal functions. When the amplitude of the sinusoid is small, several studies have established the presence of chaos in the KdV and similar equations [27–31]. The main tool used in these analyses is the Melnikov function from dynamical systems theory [32–34], whose zeros correspond to intersections between stable and unstable manifolds, and hence chaos via the Smale–Birkhoff theorem [33]. These studies do not focus on obtaining free-surface profiles theoretically, since the classical Melnikov function does not by itself relate to such profiles. However, recent theoretical developments [35,36] building on the Melnikov approach provide the proper framework for determining free-surface profiles for *general* $p(x)$, under the sole condition that p is small; indeed, Dirac delta forms are also permissible [37]. The key tool developed by Balasuriya [35] quantifies the normal and tangential motion of a stable and/or unstable manifold due to the presence of a nonautonomous perturbation. In this article, we adapt this theory to enable quantification of the free-surface profile for the KdV equation for any small p.

In obtaining the free-surface profiles for general p, we need to view the KdV system not in the (η, η_x)-phase space as is standard for autonomous or "jumping between autonomous" situations [5,20,16,17,21–26], but in the genuinely nonautonomous (η, η_x, x) phase space. This standard approach from dynamical systems is apparently not present in the KdV literature and is described in Section 2. This framework enables us to develop two classes of free-surface solutions which we classify according to dynamical systems terminology as *hyperbolic* trajectories and *homoclinic* solutions. These solution classes correspond respectively to near-uniform and near-solitary wave solutions of the KdV equation, analogous to the perturbation of a uniform stream and perturbation of a solitary wave classification of Vanden-Broeck [38]. In Section 3 we establish an analytical formula which approximates the hyperbolic (near-uniform) solution, proving moreover that for general p there is a unique hyperbolic solution. Physically, this means that there is one and only one near-uniform free-surface configuration for the steady forced KdV equation for small bottom topography. In Section 4 we establish a criterion for determining the number of homoclinic (near-solitary) solutions for a given forcing function; here 'homoclinic' means that the solution lies on both the stable and the unstable manifolds associated with the hyperbolic trajectory. This supplements theoretical results by Choi et al. [18] which characterises this number for compactly supported even p; here, we establish a tool which works for general p. We then adapt the theory of Balasuriya [35] to formulate an analytical formula which approximates each of these near-solitary waves. In Section 5, we demonstrate the excellent agreement between our theoretical formulæ and numerical solutions to the KdV equation, finding along the way several unusual looking steady solitary waves. Our ability to provide explicit analytical formulæ for the free-surface gives excellent initial guesses for our numerical KdV schemes. These can also be utilised as initial guesses in nonlinear free-surface computations beyond the weakly nonlinear (KdV) approximation [16,17,21,22,24,25].

In this work we restrict our analysis to supercritical flow with $F > 1$. The technical reason for this restriction is that in this case, the physically relevant near-uniform free-surface configuration which corresponds to $(\eta, \eta_x) = (0, 0)$ is a saddle point in the phase plane of Fig. 1(b). Such points – or more precisely their nonautonomous analogues – are structurally stable. Thus, when p is small but nonzero, a similar near-uniform solution persists; this is our hyperbolic trajectory. Moreover, the stable and unstable manifolds persist, allowing for the possibility of them intersecting to create a homoclinic solution which asymptotes to the hyperbolic solution as $x \to \pm\infty$. If we considered subcritical flow in which $F < 1$, the phase-portrait of Fig. 1(b) changes somewhat; a centre (elliptic) point now lies at the uniform free-surface location $(\eta, \eta_x) = (0, 0)$ [16]. Such an entity is not structurally stable, and thus its persistence cannot be guaranteed for p small but nonzero. Thus, our analysis for determining near-uniform or near-solitary solutions breaks down for $F < 1$. The critical case of $F = 1$, addressed in [39], is also not amenable to the present analysis since once again structural stability of the near-uniform solution is not assured.

2. Nonautonomous viewpoint

As our governing equation, we consider the integrated version of (1) given by

$$\eta_{xx} + \frac{9}{2}\eta^2 - 6(F-1)\eta = -3p(x) \qquad (2)$$