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Central extension of mapping class group via Chekhov–Fock quantization

Binbin Xu

Korea Institute for Advanced Study(KIAS), 85 Hoegiro Dongdaemun-gu, 02455 Seoul, Republic of Korea

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ABSTRACT

The central extension of mapping class groups of punctured surfaces of finite type that arises in Chekhov–Fock quantization is 12 times of the Meyer class plus the Euler classes of the punctures, which agree with the one arising in the Kashaev quantization.

1. Introduction

The quantum theory of Teichmüller spaces of punctured surfaces was developed in [1] and [2] independently, and then generalized to higher rank Lie groups and cluster algebras in [3] and [4]. The main ingredient of both constructions is Faddeev's quantum dilogarithm introduced in [5]. This theory leads to one parameter families of projective unitary representations of Ptolemy groupoids associated to ideal triangulations of punctured surfaces. These projective unitary representations moreover induce central extensions of mapping class groups of associated punctured surfaces.

The main goal of this paper is to study these central extensions arising for the Chekhov–Fock model of the quantum Teichmüller space and to identify them in terms of cohomology 2-classes of corresponding mapping class groups. This problem was first considered for the universal Teichmüller space where the mapping class group is replaced by the Thompson group. In [6], Funar and Sergiescu computed the cohomology class of central extensions of the Thompson group arising for the Chekhov–Fock model of the quantum universal Teichmüller space. In [7], Hyun Kyu Kim made further computation for the Kashaev model of the quantum universal Teichmüller space and found a different class. This suggests that the relationship between two models is subtler than thought before. Coming back to Teichmüller spaces of punctured surfaces of finite type, in [8], Funar and Kashaev computed the class arising in the Kashaev model. The present paper settles the last case open and shows that the class arising in the Chekhov–Fock model agrees with that arising in the Kashaev model.

Let V be a vector space and G be a group. A *projective representation* of G on V is a homomorphism from G to $PGL(V)$. It is well-known that a projective representation of a group gives rise to a linear representation of some central extension of the given group. More precisely, let h be a projective representation of G on V. Let $\widetilde{G}$ be a central extension of G by $\mathbb{C}^*$ which is the pullback of $GL(V) \to PGL(V)$ by h. Then one can associate a representation $\widetilde{h}$ of $\widetilde{G}$ on V, such that the following commutative diagram holds:

E-mail address: xubin@kias.re.kr.

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$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & GL(V) & \longrightarrow & PGL(V) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \widetilde{h} & & \uparrow h & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

A *reduction* $\widetilde{G}_1$ of $\widetilde{G}$ is a central extension of G by a subgroup A_1 of $\mathbb{C}^*$, such that $\widetilde{G}_1$ is a subgroup of $\widetilde{G}$ and the associated representation $\widetilde{h}_1 : \widetilde{G}_1 \to GL(V)$ is the restriction of $\widetilde{h}$. We say that $\widetilde{G}_1$ is the *minimal reduction* of $\widetilde{G}$ if $\widetilde{G}_1$ is a smallest possible reduction of $\widetilde{G}$.

Suppose that G has a presentation F/R where F is a free group and R is the normal subgroup of F generated by a set of relations. A projective representation of G on V is induced by a linear representation $\bar{h}$ of F on V such that R is sent to the center of $GL(V)$. The homomorphism $\bar{h}$ will be called an *almost linear representation* of G on V, in order to distinguish it from the projective representation.

Let Σ_g^s be an oriented compact surface of genus g with s punctures where $s > 0$. Let $\Gamma(\Sigma_g^s)$ be its mapping class group. An element of $\Gamma(\Sigma_g^s)$ is a mapping class of homeomorphisms from Σ_g^s to itself fixing the punctures setwise. Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R}^{2n})$ where n is the number of arcs in an ideal triangulation of Σ_g^s. The Chekhov–Fock quantization constructs a family of projective representations $\rho_z : \Gamma(\Sigma_g^s) \to PGL(\mathcal{H})$ depending on one parameter z with $|z| = 1$. Denote by $\widetilde{\boldsymbol{\Gamma}(\Sigma_g^s)} \to \Gamma(\Sigma_g^s)$ the central extension $\rho_z^*\eta$ where η is the central extension $GL(\mathcal{H}) \to PGL(\mathcal{H})$.

Central extensions of a group G by an abelian group A are classified, up to isomorphism, by elements of the 2-cohomology group $H^2(G, A)$. For a mapping class group $\Gamma(\Sigma_g^s)$ with $g \geq 4$ and $s \geq 2$, by the work of Harer in [9] and Korkmaz and Stipsicz in [10], we have:

$$H^2(\Gamma(\Sigma_g^s), \mathbb{Z}) = \mathbb{Z}^{1+s},$$

where the generators are given by one fourth of the Meyer signature class χ and s Euler classes e_i that are associated to s punctures p_i respectively. For $g = 3$, by the work of Sakasai in [11], the group $H^2(\Gamma(\Sigma_g^s), \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{s+1} \oplus \mathbb{Z}/2\mathbb{Z}$. For $g = 2$, it has been proved that $H^2(\Gamma(\Sigma_g^s), \mathbb{Z})$ contains the subgroup $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$ whose torsion part is generated by χ and whose free part is generated by Euler classes.

The main result in this paper is the following theorem:

Theorem 1.1. *Let $g \geq 2$ and $s \geq 4$. A minimal reduction $\widetilde{\Gamma(\Sigma_g^s)}$ of $\widetilde{\boldsymbol{\Gamma}(\Sigma_g^s)}$ is a central extension of $\Gamma(\Sigma_g^s)$ by a cyclic subgroup $A \subset \mathbb{C}^*$ generated by z^{-12}, whose cohomology class is*

$$c_{\widetilde{\Gamma(\Sigma_g^s)}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2(\Gamma(\Sigma_g^s), A).$$

The organization of this paper is as follows. In Section 2, we will recall the Chekhov–Fock quantization of the Teichmüller space of a puncture surface of finite type and describe the almost linear representation of the corresponding Ptolemy groupoid. In Section 3 we will give the proof of our main theorem.

2. Preliminaries

2.1. Ptolemy groupoids

Let $\Sigma = \Sigma_g^s$ and $\Gamma = \Gamma(\Sigma_g^s)$.

Definition 2.1. An **arc** is the homotopy class of a simple curve on Σ connecting punctures which is non homotopic to a point or a puncture of Σ.

Definition 2.2. An **ideal triangulation** of Σ is a maximal collection of distinct arcs which have pairwise disjoint representatives. A **labeled ideal triangulation** is obtained from an ideal triangulation by adding labels to its arcs.

We denote by $|\mathbf{T}(\Sigma)|$ the set of ideal triangulations of Σ and by $\mathbf{T}(\Sigma)$ the set of labeled ideal triangulations of Σ.

Let T be a labeled ideal triangulation. Let α be an arc of T which is the common boundary of two distinct ideal triangles whose union is an embedded quadrilateral in Σ.

Definition 2.3. A **flip** on α is to get a new ideal triangulation T' from T by substituting α by the other diagonal α' of the same ideal quadrilateral.