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Darboux coordinates for periodic solutions of the sinh–Gordon equation

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ABSTRACT

We study the space of periodic solutions of the elliptic sinh-Gordon equation by means of spectral data consisting of a Riemann surface Y and a divisor D and prove the existence of certain Darboux coordinates.

1. Introduction

The elliptic *sinh–Gordon equation* is given by

$$\Delta u + 2\sinh(2u) = 0, \tag{1.1}$$

where Δ is the Laplacian of $\mathbb{R}^2$ with respect to the Euclidean metric and $u : \mathbb{R}^2 \to \mathbb{C}$ is a twice partially differentiable complex-valued function.

In the present setting we only demand that u is periodic with one fixed period. After rotating the domain of definition we can assume that this period is real. This enables us to introduce the space $M^{\mathbf{p}}$ of simply periodic *Cauchy data* with fixed period $\mathbf{p} \in \mathbb{R}$ consisting of pairs $(u, u_y) \in W^{1,2}(\mathbb{R}/\mathbf{p}\mathbb{Z}) \times L^2(\mathbb{R}/\mathbf{p}\mathbb{Z})$.

In [1] the map $\Phi : (u, u_y) \mapsto (Y(u, u_y), D(u, u_y))$ was studied for finite type solutions u of the sinh-Gordon equation. Φ assigns spectral data consisting of a Riemann surface $Y(u, u_y)$ and a divisor $D(u, u_y)$ to the Cauchy data (u, u_y) of such solutions.

We will restrict to the map $(u, u_y) \mapsto D(u, u_y)$ that assigns to Cauchy data $(u, u_y) \in M^{\mathbf{p}}$ a divisor $D(u, u_y) = \sum_i (\lambda_i, \mu_i)$ on the spectral curve $Y(u, u_y)$ to potentials (u, u_y) where D has only simple points, i.e. D contains no points of higher order. Moreover, we will consider $\lambda_i,\ \mu_i$ as maps $\lambda_i,\ \mu_i : M^{\mathbf{p}} \to \mathbb{C}$.

The main goal of this paper is to prove the existence of certain Darboux coordinates for $M^{\mathbf{p}}$ and to adapt Theorem 2.8 in [2]. More precisely, we show that $(\ln \lambda_i, \ln \mu_i)$ are indeed Darboux coordinates with respect to the symplectic form $\Omega : T_{(u,u_y)}M^{\mathbf{p}} \to \mathbb{C},\ ((\delta u, \delta u_y), (\widetilde{\delta u}, \widetilde{\delta u}_y)) \mapsto \Omega((\delta u, \delta u_y), (\widetilde{\delta u}, \widetilde{\delta u}_y))$ on the tangent space $T_{(u,u_y)}M^{\mathbf{p}}$, that was introduced in [3,1].

Pöschel and Trubowitz describe Theorem 2.8 in [2] as one of the main ingredients for the investigation of the KdV equation by means of spectral theory. Since we are able to adapt this theorem to our situation, we expect that broad parts of [2] can be carried over for the sinh-Gordon equation, as well.

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2. Cauchy data and the monodromy

Let us consider the system

$$\tfrac{\partial}{\partial x}F_\lambda = F_\lambda U_\lambda, \quad \tfrac{\partial}{\partial y}F_\lambda = F_\lambda V_\lambda \quad \text{with} \quad F_\lambda(0) = \mathbb{1}$$

and

$$U_\lambda = \frac{1}{2}\begin{pmatrix} -iu_y & i\lambda^{-1}e^u + ie^{-u} \\ i\lambda e^u + ie^{-u} & iu_y \end{pmatrix}, \quad V_\lambda = \frac{1}{2}\begin{pmatrix} iu_x & -\lambda^{-1}e^u + e^{-u} \\ \lambda e^u - e^{-u} & -iu_x \end{pmatrix}. \tag{2.1}$$

The compatibility condition for this system

$$\tfrac{\partial}{\partial y}U_\lambda - \tfrac{\partial}{\partial x}V_\lambda - [U_\lambda, V_\lambda] = 0$$

holds if and only if the function $u : \mathbb{C} \to \mathbb{C}$ satisfies the **sinh–Gordon equation**

$$\Delta u + 2\sinh(2u) = 0. \tag{2.2}$$

F_λ is called **extended frame** for the pair (U_λ, V_λ).

2.1. Cauchy data (u, u_y)

We consider simply periodic solutions of (2.2) with a fixed period $\mathbf{p} \in \mathbb{C}$. After rotating the domain of definition we can assume that this period is real, i.e. $\Im(\mathbf{p}) = 0$. From now on we therefore consider simply periodic *Cauchy data* with fixed period $\mathbf{p} \in \mathbb{R}$ consisting of a pair $(u, u_y) \in W^{1,2}(\mathbb{R}/\mathbf{p}\mathbb{Z}) \times L^2(\mathbb{R}/\mathbf{p}\mathbb{Z})$.

Remark 2.1. Due to (2.1) the matrix U_λ uniquely determines the tuple (u, u_y). Vice versa, the tuple (u, u_y) determines U_λ and V_λ.

2.2. The monodromy

The central object for the following considerations is contained in

Definition 2.2. Let F_λ be an extended frame for U_λ and assume that $U_\lambda = F_\lambda^{-1}\frac{d}{dx}F_\lambda$ has period $\mathbf{p}$, i.e. $U_\lambda(x+\mathbf{p}) = U_\lambda(x)$. Then the **monodromy** of the frame F_λ with respect to the period $\mathbf{p}$ is given by

$$M_\lambda^{\mathbf{p}} := F_\lambda(x+\mathbf{p})F_\lambda^{-1}(x).$$

Since $F_\lambda(0) = \mathbb{1}$, we get

$$M_\lambda := M_\lambda^{\mathbf{p}} = F_\lambda(\mathbf{p})F_\lambda^{-1}(0) = F_\lambda(\mathbf{p}).$$

3. Spectral curve Y and divisor D

The eigenvalues and eigenlines of the monodromy M_λ are encoded in the so-called *spectral data* $(Y(u, u_y), D(u, u_y))$. We omit the dependency on (u, u_y) in the following. Let us first define the **spectral curve** Y by the eigenvalues of M_λ:

$$Y := \{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C} \mid \det(M_\lambda - \mu\mathbb{1}) = 0\}.$$

Y is a non-compact hyperelliptic Riemann surface with possible singularities and is equipped with the so-called hyperelliptic involution $\sigma : Y \to Y,\ (\lambda, \mu) \mapsto (\lambda, 1/\mu)$.

The eigenlines of the monodromy M_λ are described by normalized eigenvectors.

Lemma 3.1. *On the spectral curve Y there exist unique meromorphic maps $v(\lambda, \mu)$ and $w(\lambda, \mu)$ from Y to $\mathbb{C}^2$ such that*

(i) *For all $(\lambda, \mu) \in Y$ the value of $v(\lambda, \mu)$ is an eigenvector of M_λ with eigenvalue μ and $w(\lambda, \mu)$ is an eigenvector of M_λ^t with eigenvalue μ, i.e.*

$$M_\lambda v(\lambda, \mu) = \mu v(\lambda, \mu), \quad M_\lambda^t w(\lambda, \mu) = \mu w(\lambda, \mu).$$

(ii) *The first components of $v(\lambda, \mu)$ and $w(\lambda, \mu)$ are equal to 1, i.e. $v(\lambda, \mu) = (1, v_2(\lambda, \mu))^t$ and $w(\lambda, \mu) = (1, w_2(\lambda, \mu))^t$ on Y.*

Set $M_\lambda = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$ with holomorphic functions $a, b, c, d : \mathbb{C}^* \to \mathbb{C}$. Then we get

$$M_\lambda v(\lambda, \mu) = \mu v(\lambda, \mu) \quad \text{for} \quad v(\lambda, \mu) = \begin{pmatrix} 1 \\ \dfrac{\mu - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{c}{\mu - d} \end{pmatrix},$$