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A class of four dimensional CR submanifolds of the sphere $\mathbf{S}^6(1)$

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Dedicated to the memory of Professor Mileva Prvanović

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ABSTRACT

In this paper, we investigate four dimensional CR submanifolds of the nearly Kähler sphere $\mathbf{S}^6(1)$ that locally have a twisted product structure of special type with respect to their CR structure, and explicitly construct them by using a sphere curve. We show that four dimensional CR submanifolds with two dimensional nullity distribution belong to this class if their totally real distribution is totally geodesic and describe them.

1. Introduction

Identifying the space $\mathbb{R}^7$ with the imaginary Cayley numbers, it is possible to introduce a vector cross product $\times$ on $\mathbb{R}^7$, similar to the one in the space $\mathbb{R}^3$, which further induces an almost complex structure J on the standard unit sphere $\mathbf{S}^6(1)$ in $\mathbb{R}^7$ which makes it a nearly Kähler manifold. The group of isometries of the nearly Kähler sphere is the exceptional Lie group G_2.

Submanifolds are traditionally studied accordingly to their relation to the almost complex structure. The tangent space of almost complex submanifolds is invariant for J, and the tangent space of totally real submanifolds is mapped by J into the corresponding normal space. A natural generalization of this is the notion of CR submanifold.

A submanifold M of $\mathbf{S}^6(1)$ is called a CR submanifold, see [1], if there exists a C^∞-differential J invariant distribution $U : x \mapsto U_x \subset T_xM$ on M (i.e., $JU = U$), such that its orthogonal complement $U^\perp$ in TM is totally real ($JU^\perp \subseteq T^\perp M$), where $T^\perp M$ is the normal bundle over M in $\mathbf{S}^6(1)$. We say that M is proper if neither the almost complex, nor the totally real distribution is trivial.

Here, we are interested in four dimensional CR submanifolds. Note, that it was shown by A. Gray in [2], that there exist no four dimensional almost complex submanifolds of $\mathbf{S}^6(1)$. Hence, a four dimensional CR submanifold is automatically proper, i.e. neither U nor $U^\perp$ is zero-dimensional. CR submanifolds have been previously studied amongst others by K. Mashimo, H. Hashimoto and K. Sekigawa (see [3] and [4]).

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In [5], in particular were investigated such submanifolds in relation to the basic inequality, discovered by B.Y. Chen in [6]. There, he introduced an invariant $\delta_M = \frac{1}{2}\tau - \inf K$, of the n-dimensional submanifold of the real space form of constant sectional curvature c, where

$$\inf K(p) = \inf\{K(\pi), \text{sectional plane curvature} \quad \pi \subseteq T_p(M)\}$$

and τ is the scalar curvature, and proved that

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c.$$

Here, $\|H\|$ is the norm of the mean curvature vector H. It is interesting to investigate when the submanifold M satisfies an equality in the above inequality. Then we say that M satisfies Chen's basic equality. For such submanifolds there exists a canonical distribution defined, for the case of $\mathbf{S}^6(1)$ as the ambient space, by

$$\mathcal{D}(p) = \{X \in T_pM | 3h(X, Y) = 4\langle X, Y\rangle H, \forall Y \in T_pM\},$$

where h is the second fundamental form of M in $\mathbf{S}^6(1)$. The distribution $\mathcal{D}$ is completely integrable if the dimension of $\mathcal{D}(p)$ is constant, see also [6]. In the case of four dimensional submanifolds we have that $\dim \mathcal{D} = 2$.

In [5], the following theorem was proved.

Theorem 1. *Let M be a four dimensional minimal CR submanifold in $S^6(1)$ which satisfies Chen's equality. Then M is locally congruent with the immersion*

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = {} & (\cos x_4 \cos x_1 \cos x_2 \cos x_3, \sin x_4 \sin x_1 \cos x_2 \cos x_3, \\ & \sin 2x_4 \sin x_3 \cos x_2 + \cos 2x_4 \sin x_2, 0, -\sin x_4 \cos x_1 \cos x_2 \cos x_3, \\ & \cos x_4 \sin x_1 \cos x_2 \cos x_3, -\cos 2x_4 \sin x_3 \cos x_2 + \sin 2x_4 \sin x_2). \end{aligned} \tag{1}$$

Also, quite elementary it follows that four dimensional CR submanifolds of $S^6(1)$ cannot be totally geodesic. Therefore it is natural to investigate submanifolds that admit the nullity distribution

$$\mathcal{D}_1(p) = \{X \in T_pM | h(X, Y) = 0, \forall Y \in T_pM\}, \tag{2}$$

of the maximal possible dimension. In [7] the three dimensional CR submanifolds of the sphere $S^6(1)$ which admit the nullity distribution of the maximal dimension, which is one, were classified by constructions that start from one or two curves in $S^6(1)$. For a minimal submanifold distributions $\mathcal{D}$ and $\mathcal{D}_1$ coincide.

If the second fundamental form vanishes on the distribution $\mathcal{D}$ then it is called a totally geodesic distribution. Another interesting type of CR submanifolds are those where totally real or almost complex distribution is totally geodesic. For instance, the classification of the three dimensional CR submanifolds of $\mathbf{S}^6(1)$ with both almost complex and totally real distribution being totally geodesic was given in [8]. If ∇ is a Levi-Civita connection on (M, g) we say that a distribution $\mathcal{D}$ on M is totally umbilical if there exists a vector field $H \in \mathcal{D}^\perp$ such that $g(\nabla_X Y, Z) = g(X, Y)g(H, Z)$ for $X, Y \in \mathcal{D}, Z \in \mathcal{D}^\perp$.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, $M = M_1 \times M_2$, and $\pi_i : M \to M_i$ the canonical projections. For the tangent vector fields X, Y of the manifold M, we denote by $g(X, Y) = g_1(d\pi_1 X, d\pi_1 Y) + \lambda^2 g_2(d\pi_2 X, d\pi_2 Y)$, where $\lambda^2 > 0$ is a function on M. Then, g is a metric on the manifold M, and we call (M, g) the twisted product manifold, which we also denote by $M_1 \times_{\lambda^2} M_2$. The canonical foliations K_1 and K_2 corresponding to M_1 and M_2 intersect perpendicularly everywhere. Moreover K_1 is a totally geodesic and K_2 a totally umbilical foliation.

We recall the splitting theorem for a Riemannian manifold (M, g) given in [9], stating that if M admits two complementary foliations K_1, K_2 whose leaves intersect perpendicularly such that the leaves of K_1 are totally geodesics and the leaves of K_2 are totally umbilical, then (M, g) is locally isometric to a twisted product $M_1 \times_{\lambda^2} M_2$ such that M_1 and M_2 are leaves of K_1 and K_2, respectively.

It was shown in [10] that there do not exist (doubly) twisted product CR submanifolds in the nearly Kähler manifolds with components corresponding to almost complex and totally real distribution. Here we consider another kind of possible decomposition of a four dimensional CR submanifold of $\mathbf{S}^6(1)$ and prove the following.

Theorem 2. *Let $M = N \times_{\lambda^2} I$ be a four dimensional CR submanifold of sphere $S^6(1)$ with totally real distribution $U^\perp$, which is also a twisted product of a totally real three dimensional submanifold N and an interval I with tangent vector field T. If $U^\perp$ is a totally geodesic distribution and JT is tangent to M, then, under restriction that* $\dim U^\perp \cap \mathcal{D}_1 = const.$, *we have that N is a part of a totally geodesic sphere $\mathbf{S}^3$, the nullity distribution of M is at least one dimensional and M is locally congruent to the immersion*

$$F(y_1, y_2, y_3, y_4, s) = y_1\gamma(s) + y_2 A_3(s) + y_3 A_3 \times \gamma'(s) - y_4(\gamma \times \gamma')(s), \tag{3}$$

where γ is an unit length sphere curve, A_3 is a unit length vector field along γ orthogonal to γ, γ' and $\gamma \times \gamma'$ and $y_1^2+y_2^2+y_3^2+y_4^2 = 1$. Moreover we have

(1) if $\dim U^\perp \cap \mathcal{D}_1 = 1$, *$M$ is obtained for $A_3' \times A_3$ orthogonal to γ and $\gamma \times \gamma'$, and* $\dim \mathcal{D}_1 = 2$ *for $A_3 - \langle A_3', \gamma'\rangle\gamma$ orthogonal to $\gamma'' \times \gamma'$.*