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A Hamilton–Jacobi Theory for general dynamical systems and integrability by quadratures in symplectic and Poisson manifolds

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ABSTRACT

In this paper we develop, in a geometric framework, a Hamilton–Jacobi Theory for general dynamical systems. Such a theory contains the classical theory for Hamiltonian systems on a cotangent bundle and recent developments in the framework of general symplectic, Poisson and almost-Poisson manifolds (including some approaches to a Hamilton–Jacobi Theory for nonholonomic systems). Given a dynamical system, we show that every complete solution of its related Hamilton–Jacobi Equation (HJE) gives rise to a set of first integrals, and vice versa. From that, and in the context of symplectic and Poisson manifolds, a deep connection between the HJE and the (non)commutative integrability notion, and consequently the integrability by quadratures, is established. Moreover, in the same context, we find conditions on the complete solutions of the HJE that also ensures integrability by quadratures, but they are weaker than those related to the (non)commutative integrability. Examples are developed along all the paper in order to illustrate the theoretical results.

1. Introduction

In the context of Classical Mechanics, the *standard* or *classical* (time-independent) Hamilton–Jacobi Theory was designed to construct, for a given Hamiltonian system on a cotangent bundle T^*Q, local coordinates such that the Hamilton equations expressed on these coordinates adopt a very simple form. Here, by *simple* we mean that such equations can be integrated by *quadratures* (i.e. its solutions can be given in terms of primitives and inverses of known functions). The fundamental tool of the theory is the so-called (time-independent) *Hamilton–Jacobi equation (HJE)* for a Hamiltonian function $H : T^*Q \to \mathbb{R}$. The problem is to find a function W on Q, known as the *characteristic Hamilton function*, such that the equation (the classical HJE)

$$H\left(q, \frac{\partial W}{\partial q}\right) = \text{constant}$$

is satisfied (see for example [1]). When we are given a family of solutions $\{W_\lambda\}_{\lambda\in\Lambda}$ such that the square matrix $\partial^2 W_\lambda/\partial q \partial\lambda$ is non-degenerated, with Λ an open subset of $\mathbb{R}^{\dim Q}$, the above mentioned coordinates can be constructed. More precisely,

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a type 2 canonical transformation (see [2]) can be defined from the functions W_λ's such that the equations of motion of the system become, under such a transformation, simple enough to be solved by quadratures.

In modern terms (see [3]), the classical HJE reads $d(H \circ \sigma) = 0$, and its unknown is a closed 1-form $\sigma : Q \to T^*Q$. If a solution σ is given for a Hamiltonian function H, the problem of finding the integral curves of its Hamiltonian vector field X_H (w.r.t. the canonical symplectic structure on T^*Q), with initial conditions along the image of σ, reduces to find the integral curves of the vector field $X_H^\sigma := (\pi_Q)_* \circ X_H \circ \sigma$ on Q, where $\pi_Q : T^*Q \to Q$ is the cotangent fibration. If one wants to find all the integral curves of X_H, i.e. the solutions of the equations of motion of the system for every initial condition, it is necessary to introduce the notion of a *complete solution*: a surjective local diffeomorphism $\Sigma : Q \times \Lambda \to T^*Q$, with Λ a manifold, such that $\sigma_\lambda := \Sigma(\lambda, \cdot)$ is a solution of the classical HJE for each $\lambda \in \Lambda$. In terms of Σ, around every point of T^*Q, a local coordinate chart can be constructed such that the equations of motion are easily solved. We can see that through the well-known connection between the classical Hamilton–Jacobi Theory and the notion of *commutative integrability* (see, for example, [3]). From every complete solution $\Sigma : Q \times \Lambda \to T^*Q$, we can construct (unless locally) a Lagrangian fibration $F : T^*Q \to \Lambda$ (transverse to π_Q) such that the image of X_H lives inside $\operatorname{Ker} F_*$. In other terms, if Λ is an open subset of $\mathbb{R}^n$ (with $n = \dim Q$), we can build up n first integrals $f_i : T^*Q \to \mathbb{R}$ (the components of F) which are independent and in involution with respect to the canonical Poisson bracket. Conversely, from every Lagrangian fibration $F : T^*Q \to \Lambda$ (transverse to π_Q) such that $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$, a complete solution of the HJE can be constructed (unless locally). We recall that, if such a fibration is given for a Hamiltonian system, the Arnold–Liouville Theorem [4] establishes that the system in question is integrable by quadratures.

In the last few years, several generalized versions of the classical HJE have been developed for Hamiltonian systems on different contexts: on general symplectic, Poisson and almost-Poisson manifolds, and also on Lie algebroids over vector bundles. The resulting Hamilton–Jacobi theories were applied to nonholonomic systems, time-dependent Hamiltonian systems, reduced systems by symmetries and systems with external forces [5,3,6–8]. In all of them, a fibration $\Pi : M \to N$ (i.e. a surjective submersion) is defined on the phase space M of each system, the solutions of the generalized HJE are sections $\sigma : N \to M$ of such a fibration and the complete solutions are local diffeomorphisms $\Sigma : N \times \Lambda \to M$ such that $\sigma_\lambda := \Sigma(\lambda, \cdot)$ is a solution of the HJE for each $\lambda \in \Lambda$. This clearly extends the classical situation, where the involved fibration is the cotangent projection $\pi_Q : T^*Q \to Q$ of a manifold Q. Unfortunately, no connection between complete solutions and some kind of *exact solvability* (as the integrability by quadratures) has been given for any of those generalized versions of the HJE (see, for instance, [9]).

This paper is the first of a series of papers in which we shall further extend the previously mentioned Hamilton–Jacobi theories to general dynamical systems on a fibered phase space. We are mainly interested in the connection between complete solutions and exact solvability. In the present paper, as a first step, we shall focus on (time-independent) Hamiltonian systems on general Poisson manifolds. In forthcoming papers, we shall address the case of Hamiltonian systems with external forces (including Hamiltonian systems with constraints) and time-dependent Hamiltonian systems.

One of the contribution of this paper is to show, in the context of general dynamical systems, that there exists a *duality* between complete solutions and first integrals, extending similar results that appear in the literature. This enable us to establish, in the particular context of Hamiltonian systems on Poisson manifolds, a deep connection between *(non)commutative integrability* and certain subclasses of complete solutions.

Recall that a Hamiltonian system on a Poisson manifold M, with Hamiltonian function H, is a *noncommutative integrable* system (see [10,11] for symplectic manifolds and [12,13] for Poisson ones) if a fibration $F : M \to \Lambda$ such that:

1. $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$ (F defines first integrals for the system),
2. $\operatorname{Ker} F_* \subset (\operatorname{Ker} F_*)^\perp$, i.e. F is isotropic,
3. $(\operatorname{Ker} F_*)^\perp$ is integrable (F has a polar),

can be exhibited. In particular, the system is commutative integrable if in addition $\operatorname{Ker} F_* = (\operatorname{Ker} F_*)^\perp$, i.e. F is Lagrangian, as we have said above. By $\perp$ we are denoting the Poisson orthogonal. To be more precise, if Ξ is the Poisson bi-vector on M and $\Xi^\sharp : T^*M \to TM$ is its related linear bundle map, then $(\operatorname{Ker} F_*)^\perp := \Xi^\sharp((\operatorname{Ker} F_*)^0)$. All of these systems, among other things, are integrable by quadratures.

Another contribution of the paper (which can be seen as a first application of our extended Hamilton–Jacobi Theory) is to show that conditions 1 and 2 listed above are enough in order to ensure integrability by quadratures of Hamiltonian systems on Poisson manifolds. That is to say, the integrability of $(\operatorname{Ker} F_*)^\perp$ is not needed for that purpose. Moreover, we show that condition 2 can be replaced by a weaker one: $\operatorname{Ker} F_* \cap \operatorname{Im} \Xi^\sharp \subset (\operatorname{Ker} F_*)^\perp$, which we call *weak isotropy* (together some regularity assumptions about the symplectic leaves of Ξ). It is worth mentioning that the proof of this result was done mainly in terms of complete solutions (instead of first integrals). We think that, in spite of the duality between complete solutions and first integrals, it would have been very hard to anticipate the mentioned result by working with first integrals only.

The paper is organized as follows. In Section 2, given a dynamical system (M, X) equipped with a fibration $\Pi : M \to N$, being M and N smooth manifolds and X a vector field on M, we introduce the notion of Π-Hamilton–Jacobi equation (Π-HJE) for (M, X). The unknown of such an equation is a section $\sigma : N \to M$ of Π. The Π-HJE is defined in such a way that, if σ is a solution, then the vector field X restricts to the closed submanifold $\operatorname{Im} \sigma \subset M$ determined by the image of σ. For the case in which M is a Poisson manifold and X is a Hamiltonian vector field, we give characterizations of the corresponding Π-HJEs in terms of (co)isotropic fibrations. In particular, when M is a symplectic manifold, we recover the well-known results for the