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On power series expansions of the S-resolvent operator and the Taylor formula

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ABSTRACT

The S-functional calculus is based on the theory of slice hyperholomorphic functions and it defines functions of n-tuples of not necessarily commuting operators or of quaternionic operators. This calculus relays on the notion of S-spectrum and of S-resolvent operator. Since most of the properties that hold for the Riesz–Dunford functional calculus extend to the S-functional calculus, it can be considered its non commutative version. In this paper we show that the Taylor formula of the Riesz–Dunford functional calculus can be generalized to the S-functional calculus. The proof is not a trivial extension of the classical case because there are several obstructions due to the non commutativity of the setting in which we work that have to be overcome. To prove the Taylor formula we need to introduce a new series expansion of the S-resolvent operators associated to the sum of two n-tuples of operators. This result is a crucial step in the proof of our main results, but it is also of independent interest because it gives a new series expansion for the S-resolvent operators. This paper is addressed to researchers working in operator theory and in hypercomplex analysis.

1. Introduction

The theory of slice hyperholomorphic functions, introduced in the last decade, turned out to be an important tool in operator theory. We call slice hyperholomorphic functions two classes of hyperholomorphic functions: the slice monogenic and the slice regular ones. Slice monogenic functions are defined on subsets of the Euclidean space $\mathbb{R}^{n+1}$ and have values in the real Clifford algebra $\mathbb{R}_n$, while slice regular functions are defined on subsets of the quaternions and are quaternion-valued. These two sets of functions are in the kernel of a suitable Cauchy–Riemann operator and so they constitute the class of hyperholomorphic functions for which the S-functional calculus is defined. This class of functions plays the analogue role of the holomorphic functions for the Riesz–Dunford functional calculus.

Precisely, when the S-functional calculus is based on slice monogenic functions, it defines functions of n-tuples of not necessarily commuting operators, see [1–4], its commutative version is in [5], while when we consider slice regular functions the S-functional calculus defines functions of quaternionic operators (the quaternionic functional calculus), see [2,6,7]. To define the S-functional calculus a new notion of spectrum for n-tuples of operators and for quaternionic operators has been introduced. It is called S-spectrum and it naturally arises from the Cauchy formula of slice hyperholomorphic functions. For a global picture of the function theory and of the S-functional calculus see the book [8].

The discovery of the S-spectrum and the S-functional calculus allowed mathematicians to generalize many of the fundamental techniques in operator theory to the non commutative setting. A continuous functional calculus for quaternionic

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bounded linear normal operators in a Hilbert space, which is based on the notion of S-spectrum, has been introduced in [9]. A theory of quaternionic linear Schatten-class operators has been developed in [10] and the Philips-functional calculus for generators of strongly continuous groups has been generalized to the quaternionic setting in [11]. Furthermore, it was possible to define fractional powers of quaternionic linear operators in [12].

On the S-spectrum is also based a spectral theorem for quaternionic unitary operators see [13] whose proof uses a theorem of Herglotz in [14]. For the spectral theorem of compact quaternionic normal operators see [15]. For the spectral theorem based on the S-spectrum for quaternionic bounded or unbounded normal operators see the paper [13].

Moreover, we point out that there exists a relation to the monogenic functional calculus: the so-called $\mathcal{F}$-functional calculus defines monogenic functions of an operator via the theory of slice hyperholomorphic functions and Fueter's theorem in integral form [16,17].

In this paper we will formulate our results for the case of n-tuples of non commuting operators, but what we prove here holds also for the quaternionic functional calculus.

Let us recall the Taylor formula for the Riesz–Dunford functional calculus, see [18] p. 590 for more details, and then we show its generalization to our setting.

Let A and N be bounded commuting operators on a complex Banach space X. Let f be an analytic function on a domain $D \subset \mathbb{C}$ that includes the spectrum $\sigma(A)$ of A and every point within a distance of $\sigma(A)$ not greater than some positive number ε. Suppose that the spectrum $\sigma(N)$ of N lies within the open circle of radius ε about the origin.

Then f is analytic on a neighborhood of $\sigma(A+N)$ and the series

$$f(A+N) = \sum_{m=0}^{\infty} \frac{f^{(m)}(A)}{m!} N^m,$$

converges in the uniform operator topology.

One of the main points in the proof of this theorem is the series expansion of the resolvent operator $R(\lambda, A+N) := (\lambda I - A - N)^{-1}$ in terms of the resolvent of A and of powers of N. Precisely the series

$$V_\lambda(A,N) := \sum_{m=0}^{\infty} R(\lambda, A)^{m+1} N^m \tag{1.1}$$

converges uniformly for λ in a set whose minimum distance from $\sigma(A)$ is greater than ε, and since A and N commute it is easy to verify that

$$V_\lambda(A,N)(\lambda I - A - N) = (\lambda I - A - N)V_\lambda(A,N) = \mathcal{I},$$

so $V_\lambda(A,N)$ is the required power series expansion for the resolvent operator of $A+N$.

In the case of n-tuples of not necessarily commuting operators we will see that the analogue of (1.1) is not so easy to find, but despite this difficulty we are still able to prove the Taylor formula.

Let $T_j : V \to V$, $j = 0, 1, \dots, n$ be bounded $\mathbb{R}$-linear operators defined on a real Banach space V. Associated with this $(n+1)$-tuple of operators we define the paravector operator T as $T := T_0 + e_1 T_1 + \dots + T_n$ where e_j, $j = 1, \dots, n$ are the units of the real Clifford algebra $\mathbb{R}_n$. In this paper it will always consider bounded paravector operators.

The S-spectrum of T is

$$\sigma_S(T) := \{s \in \mathbb{R}^{n+1} \ : \ T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \textit{ is not invertible}\}.$$

The S-resolvent set is defined as $\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T)$, the left S-resolvent operator is

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \quad s \in \rho_S(T)$$

and the right S-resolvent operator is

$$S_R^{-1}(s,T) := -(T - \bar{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}, \quad s \in \rho_S(T).$$

In this case the analogue of the power series (1.1) requires a new series expansion for the S-resolvent operators which involves, for $n \geq 0$, the operator

$$S_L^{-n}(s,T) := (T^2 - 2s_0 T + |s|^2\mathcal{I})^{-n}(\bar{s}\mathcal{I} - T)^{\star_\ell n}, \quad s \in \rho_S(T)$$

where the Newton binomial $(\bar{s}\mathcal{I} - T)^{\star_\ell n}$ is computed allowing $\bar{s}$ and T to commute in a sense explained in the sequel. When we assume that the bounded paravector operators T and N commute and are such that $\sigma_S(N) \subset B_\varepsilon(0)$, then, for any $s \in \rho_S(T)$ with $\mathrm{dist}(s, \sigma_S(T)) > \varepsilon$, we obtain the following expansions for the S-resolvent operators:

$$S_L^{-1}(s, T+N) = \sum_{n=0}^{\infty} N^n S_L^{-(n+1)}(s,T) \quad \text{and} \quad S_R^{-1}(s, T+N) = \sum_{n=0}^{\infty} S_R^{-(n+1)}(s,T) N^n.$$

Moreover, these series converge uniformly on any set C with $\mathrm{dist}(C, \sigma_S(T)) > \varepsilon$.

A crucial fact to obtain these series expansions is to prove the invertibility of the term

$$(T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}$$

under the required conditions on the operators T and N.