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Vertex algebras associated to a class of Lie algebras of Krichever–Novikov type

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ABSTRACT

In this paper, we study a class of infinite-dimensional Lie algebras generalizing the constant r-term Krichever–Novikov type algebras. We associate vertex $\mathbb{C}((z))$-algebras and their type zero modules to representations of these Lie algebras. An equivalence between the category of restricted modules for the Lie algebras and the category of type zero modules for the vertex $\mathbb{C}((z))$-algebras is established.

1. Introduction

The Virasoro algebra is a central extension of the Lie algebra of meromorphic vector fields on a genus zero Riemann surface that are holomorphic except at two marked points. In 1987, Krichever and Novikov introduced and studied certain classes of infinite dimensional Lie algebras to extend the Virasoro algebra to Riemann surfaces of higher genus with two marked points in [1–3]. The multi-point generalizations were given by Schlichenmaier in [4,5]. In general, there is no non-trivial grading anymore for KN algebras. Krichever and Novikov proposed the following concept of almost-grading for Lie algebras. Let $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ be a Lie algebra, which is a vector space direct sum. Then $\mathfrak{g}$ is called an almost-graded Lie algebra if there exist constants R, S such that

$$[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \bigoplus_{i=R}^{S} \mathfrak{g}_{m+n+i} \quad \text{for } m, n \in \mathbb{Z}.$$

For more generalizations of KN algebras, we refer readers to Schlichenmaier's extensive survey paper [6].

From a purely algebraic point of view, Fairlie, Fletcher and Nuyts in [7] introduced a family of the simplest almost-graded Lie algebras $\mathfrak{e}$, named the constant r-term Krichever–Novikov type algebras, with a basis $\{e_n \mid n \in \mathbb{Z}\}$ satisfying

$$[e_m, e_n] = \sum_{i=0}^{N} r_i(m-n)e_{m+n+i} \quad \text{for } m, n \in \mathbb{Z},$$

where $r_i \in \mathbb{C}$ and N is a fixed natural number. A minimal set of defining generators $\mathfrak{e}$ was given by Nuyts and Platten in [8]. In [9], Xu presented an explicit formula of 2-cocycle χ of the Lie algebra $\mathfrak{e}$ and constructed some representations of the centrally extended constant r-term Krichever–Novikov algebra by certain vertex operator approach.

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On the other hand, a class of nonassociative algebras, named Novikov algebras, were first introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus (cf. [10,11]). By the affinization of Novikov algebras, Balinsky and Novikov presented an interesting method of constructing infinite-dimensional $\mathbb{Z}$-graded Lie algebras, which includes as special cases the Virasoro algebra (corresponding to the 1-dimensional field) as well as many Virasoro-type Lie algebras (cf. [12–14] and references therein).

Motivated by the Balinsky and Novikov's construction, we start with a Novikov algebra $\mathfrak{n}$ equipped with a symmetric invariant bilinear form and a polynomial $p(t) \in \mathbb{C}[t]$ to construct an infinite-dimensional almost-graded Lie algebra $\hat{\mathfrak{n}}_p$ with underlying vector space $\hat{\mathfrak{n}}_p = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}] \bigoplus \mathbb{C}\mathbf{c}$, which includes as special cases the centrally extended constant r-term Krichever–Novikov algebras, the Lie algebras of generalized Block type, as well as the algebra of area-preserving symplectic diffeomorphisms of a two-dimensional cylinder (cf. [15,16]).

It is well known (cf. [17,18]) that the Virasoro algebra Vir gives rise to a particular example of vertex algebras. Moreover there is an isomorphism between the category of restricted Vir-modules of central charge ℓ and the category of $M_{\mathrm{Vir}}(\ell, 0)$-modules, where $M_{Vir}(\ell, 0)$ is the vertex algebra arising from the vacuum module of Vir. As for the Lie algebra $\hat{\mathfrak{n}}_p$, the situation is different in an essential way. Since $\hat{\mathfrak{n}}_p$ does not admit nontrivial vacuum modules, we have to find an alternative. Motivated by the conceptual construction of vertex algebras and their modules in [19], we consider restricted modules W for $\hat{\mathfrak{n}}_p$ in the sense that all the generating functions of $\hat{\mathfrak{n}}_p$ lie in $\mathrm{Hom}(W, W((x)))$. For any restricted $\hat{\mathfrak{n}}_p$-module W, the generating functions form a local subset of $\mathrm{Hom}(W, W((x)))$ and it follows from [19] that the generating functions generate a vertex algebra V_W with W as a canonical module. However, the vertex algebra V_W is not a $\hat{\mathfrak{n}}_p$-module under the canonical action, unlike the case for the Virasoro algebra Vir. To a certain extent, this phenomenon is similar to the case for elliptic affine Lie algebras $\hat{\mathfrak{g}}_e$ where the generating functions for $\hat{\mathfrak{g}}_e$ on a restricted module generate a vertex algebra which under the canonical action is a module for another Lie algebra $\check{\mathfrak{g}}_e$, but not for the elliptic affine Lie algebras $\hat{\mathfrak{g}}_e$ (cf. [20]).

We then introduce another Lie algebra $\check{\mathfrak{n}}_p$ over the field $\mathbb{C}((z))$, where z is a formal variable. To a certain extent, Lie algebra $\check{\mathfrak{n}}_p$ can be regarded as a deformation of $\hat{\mathfrak{n}}_p$. Viewed as a Lie algebra over $\mathbb{C}$, the Lie algebra $\check{\mathfrak{n}}_p$ admits vacuum modules. For every complex number ℓ, using an induced module construction we construct a universal vacuum module $V_{\check{\mathfrak{n}}_p}(\ell, 0)$, and we prove that there exists a canonical vertex algebra structure on this vacuum module. Though $V_{\check{\mathfrak{n}}_p}(\ell, 0)$ is also a $\mathbb{C}((z))$-module, it is not a vertex algebra over the field $\mathbb{C}((z))$. The vertex algebra structure and the $\mathbb{C}((z))$-module structure on $V_{\check{\mathfrak{n}}_p}(\ell, 0)$ are encoded into a structure of a vertex $\mathbb{C}((z))$-algebra. The notion of vertex $\mathbb{C}((z))$-algebra is a special case of the notion of weak quantum vertex $\mathbb{C}((z))$-algebra, which was introduced and studied in [21] (cf. [22–25] for more details on quantum vertex algebras). Here we use those results on quantum vertex $\mathbb{C}((z))$-algebras in an essential way. As for the Lie algebra $\hat{\mathfrak{n}}_p$, we then prove that for any complex number ℓ, the category of restricted $\hat{\mathfrak{n}}_p$-modules of level ℓ is canonically isomorphic to the category of type zero modules for a certain vertex $\mathbb{C}((z))$-algebra $V_{\check{\mathfrak{n}}_p}(\ell, 0)$.

This paper is organized as follows: In Section 2, for each polynomial $p(t)$ and a Novikov algebra $\mathfrak{n}$, we shall construct a class of Lie algebras $\hat{\mathfrak{n}}_p$ over $\mathbb{C}$, which generalizes the Virasoro algebra of the constant r-term Krichever–Novikov type in a certain way. In Section 3, we construct another closely related Lie algebra $\check{\mathfrak{n}}_p$ over the field $\mathbb{C}((z))$. In Section 4, We associate vertex $\mathbb{C}((z))$-algebras and their type zero modules to $\hat{\mathfrak{n}}_p$. We prove that a restricted $\hat{\mathfrak{n}}_p$-module structure of level ℓ is equivalent to a type zero $V_{\check{\mathfrak{n}}_p}(\ell, 0)$-module.

Throughout this paper all vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers, unless it is specified otherwise. We shall use the formal variable notations and conventions as established in [26] (cf. [18]). For any formal variable z, let $\mathbb{C}((z))$ denote the field of lower truncated formal Laurent series.

2. Generalized Krichever–Novikov algebras of constant r-term

In this section, let $\mathbb{F}$ be an extension field of the field of complex numbers $\mathbb{C}$. For each polynomial $p(t) \in \mathbb{F}[t]$ and a Novikov algebra $\mathfrak{n}$, we shall construct a Lie algebra over $\mathbb{F}$, which generalizes the Virasoro algebra of the constant r-term Krichever–Novikov type in a certain way (cf. [8,9]).

Now we recall the definition of Novikov algebras.

Definition 2.1 (*cf. [10,11,27]*)**.** A Novikov algebra $\mathfrak{n}$ is a vector space over $\mathbb{F}$ with a bilinear multiplicative operation $\circ$ satisfying

$$(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ b), \tag{1}$$

$$(a \circ b) \circ c = (a \circ c) \circ b \tag{2}$$

for $a, b, c \in \mathfrak{n}$.

Proposition 2.2. *Let $\mathfrak{n}$ be a Novikov algebra and $p(t) \in \mathbb{F}[t]$. Set*

$$L(\mathfrak{n}, p(t), \mathbb{F}) = \mathfrak{n} \otimes \mathbb{F}[t, t^{-1}]. \tag{3}$$

Define a bilinear operation on $L(\mathfrak{n}, p(t), \mathbb{F})$ by

$$[a \otimes f(t), b \otimes g(t)] = (a \circ b) \otimes f'(t)g(t)p(t) - (b \circ a) \otimes f(t)g'(t)p(t) \tag{4}$$

for $a, b \in \mathfrak{n}, f(t), g(t) \in \mathbb{F}[t, t^{-1}]$. Then $L(\mathfrak{n}, p(t), \mathbb{F})$ is a Lie algebra over $\mathbb{F}$.