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On the Kähler metrics over $\mathrm{Sym}^d(X)$

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ABSTRACT

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. For each $d < \eta(X)$, where $\eta(X)$ is the gonality of X, the symmetric product $\mathrm{Sym}^d(X)$ embeds into $\mathrm{Pic}^d(X)$ by sending an effective divisor of degree d to the corresponding holomorphic line bundle. Therefore, the restriction of the flat Kähler metric on $\mathrm{Pic}^d(X)$ is a Kähler metric on $\mathrm{Sym}^d(X)$. We investigate this Kähler metric on $\mathrm{Sym}^d(X)$. In particular, we estimate it is Bergman kernel. We also prove that any holomorphic automorphism of $\mathrm{Sym}^d(X)$ is an isometry.

1. Introduction

Symmetric products of Riemann surfaces were studied by Macdonald [1]; he explicitly computed their cohomologies. Interests on these varieties revived when it was realized that they constitute examples of vortex moduli spaces [2–4]. One of the questions was to compute the volume, which was resolved in a series of papers [5–7]; see also [8] for Kähler structure on vortex moduli spaces.

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$, and let $\eta(X)$ denote the gonality of X (this means that X admits a nonconstant holomorphic map to $\mathbb{CP}^1$ of degree $\eta(X)$ and it does not have any smaller degree nonconstant holomorphic map to $\mathbb{CP}^1$). Take any integer $1 \leq d < \eta(X)$. Let

$$\varphi \,:\, \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X)$$

be the map from the symmetric product that sends any $\{x_1, \ldots, x_d\}$ to the holomorphic line bundle $\mathcal{O}_X(x_1 + \cdots + x_d)$. We prove that φ is an embedding.

The natural inner product on $H^0(X,\, K_X)$, where $K_X \longrightarrow X$ is the holomorphic cotangent bundle, produces a flat Kähler metric on $\mathrm{Pic}^d(X)$. It is natural to construct a metric on $\mathrm{Sym}^d(X)$ by pulling back the flat metric using the embedding φ; see [9,10] (especially [10, p. 1137, (1.2)], [10, § 7]). Our aim here is to study this metric on $\mathrm{Sym}^d(X)$. We prove that any

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holomorphic automorphism of $\mathrm{Sym}^d(X)$ is in fact an isometry. Our main result is estimation of the Bergman kernel of the metric.

Classically, the Bergman kernel which is the reproducing kernel for L^2-holomorphic functions has been extensively studied in complex analysis. The generalization of the Bergman kernel to complex manifolds as the kernel for the projection onto the space of harmonic (p, q)-forms with L^2-coefficients carries the information on the algebraic and geometric structures of the underlying manifolds.

Using results from [11] and [12], we derive the following estimate for $\mathcal{B}_X(z)$, the Bergman kernel associated to the Riemann surface X:

$$\mathcal{B}_X(z) \leq \frac{48}{\pi} + \frac{4}{3\pi\, \sinh^2(r_X/4)},$$

where r_X denotes the injectivity radius of X.

We also study the above estimate for admissible sequences of compact hyperbolic Riemann surfaces. Our estimates are optimal, and these estimates continue to hold true for any compact hyperbolic Riemann surface.

2. Comparison of Kähler metrics

In this section, we introduce the hyperbolic and canonical metrics defined on a compact hyperbolic Riemann surface. Furthermore, we introduce the Bergman kernel, and derive estimates for it. We then extend these estimates to admissible sequences of compact hyperbolic Riemann surfaces.

2.1. Canonical and hyperbolic metrics

Let X be a compact, connected Riemann surface of genus g, with $g > 1$. Let

$$\mathbb{H} := \{z = x + \sqrt{-1}y \in \mathbb{C} \mid y > 0\}$$

be the upper half-plane. Using the uniformization theorem X can be realized as the quotient space $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a torsionfree cocompact Fuchsian subgroup acting on $\mathbb{H}$, via fractional linear transformations.

Locally, we identify X with its universal cover $\mathbb{H}$ using the covering map $\mathbb{H} \longrightarrow X$.

The holomorphic cotangent bundle on X will be denoted by K_X. Let

$$\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$$

be the Jacobian variety that parametrizes all the (holomorphic) isomorphism classes of topologically trivial holomorphic line bundles on X. It is equipped with a flat Kähler metric g_J given by the Hermitian structure on $H^0(X, K_X)$ defined by

$$(\alpha, \beta) \longmapsto \frac{\sqrt{-1}}{2}\int_X \alpha \wedge \overline{\beta}. \tag{2.1}$$

Fix a base point $x_0 \in X$. Let

$$\mathrm{AJ}_X : X \longrightarrow \mathrm{Jac}(X)$$

be the Abel-Jacobi map that sends any $x \in X$ to the holomorphic line bundle on X of degree zero given by the divisor $x - x_0$. It is a holomorphic embedding of X. The pulled back Kähler metric $\mathrm{AJ}_X^* g_J$ on X is called the *canonical metric*. The $(1, 1)$-form on X associated to the canonical metric is denoted by μ_X^{can}.

The canonical metric has the following alternate description. Let $S_2(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight-2 with respect to Γ. Let $\{f_1, \ldots, f_g\}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. Then, the $(1, 1)$-form $\mu_X^{\mathrm{can}}(z)$ corresponding to the canonical metric of X is given by

$$\mu_X^{\mathrm{can}}(z) := \frac{\sqrt{-1}}{2g}\sum_{j=1}^{g}\left|f_j(z)\right|^2 dz \wedge d\overline{z}. \tag{2.2}$$

The volume of X with respect to the canonical metric is one.

Consider the hyperbolic metric of X, which is compatible with the complex structure on X and has constant negative curvature -1. We denote by μ_X^{hyp} the $(1, 1)$–form on X corresponding to it. The hyperbolic form on $\mathbb{H}$ is given by

$$\frac{\sqrt{-1}}{2} \cdot \frac{dz \wedge d\overline{z}}{\mathrm{Im}(z)^2}.$$