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Fourier theory and C^*-algebras

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ABSTRACT

We discuss a number of results concerning the Fourier series of elements in reduced twisted group C^*-algebras of discrete groups, and, more generally, in reduced crossed products associated to twisted actions of discrete groups on unital C^*-algebras. A major part of the article gives a review of our previous work on this topic, but some new results are also included.

A Banach space X is said to have the metric approximation property (MAP) if the identity map on X is a point-norm limit of finite-rank linear contractions. As a result of work of M.D. Choi, E. Effros, E. Kirchberg and others in the 1970s, it was known that a C^*-algebra A is nuclear if and only if it has the completely positive approximation property (CPAP), i.e., the identity map on A is a point-norm limit of completely positive finite-rank linear contractions. In particular, A has the MAP whenever it is nuclear, and it was believed that the converse should also be true. It came therefore as a surprise when U. Haagerup was able to show in [1] that the reduced group C^*-algebra $C_r^*(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ is an example of a nonnuclear C^*-algebra having the MAP. In the course of the proof of this result, Haagerup actually showed the following facts, that for different reasons have exerted a lasting influence on the subsequent development of noncommutative harmonic analysis.

- Let $|\cdot|$ denote the word length function on $\mathbb{F}_2 = \langle a, b\rangle$ w.r.t. $S = \{a, b, a^{-1}, b^{-1}\}$. Then $|\cdot|$ is proper and negative definite. Equivalently, using Schoenberg's theorem, the functions on $\mathbb{F}_2$ given by $\psi_t(g) = e^{-t|g|}$ are vanishing at infinity and positive definite for every $t > 0$. Since ψ_t converges pointwise to 1 as $t \to 0$, this means, using more recent terminology [2], that $\mathbb{F}_2$ has the *Haagerup property*.

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- If $f \in C_c(\mathbb{F}_2)$, that is, if f is complex function on $\mathbb{F}_2$ having finite support, and $\lambda(f)$ denotes the associated left convolution operator acting on $\ell^2(\mathbb{F}_2)$, then the operator norm of $\lambda(f)$ satisfies

$$\|\lambda(f)\| \leq 2\left(\sum_{g\in\mathbb{F}_2} |f(g)|^2(1+|g|)^4\right)^{1/2} \quad \left(=2\,\|f(1+|\cdot|)^2\|_2\right).$$

  Thus $\mathbb{F}_2$ has the *rapid decay property* (RD) in the sense of P. Jolissaint [3].
- If $\varphi : \mathbb{F}_2 \to \mathbb{C}$ is such that $K := \sup_{g\in\mathbb{F}_2} |\varphi(g)|(1+|g|)^2 < \infty$, then we have

$$\|\lambda(\varphi f)\| \leq 2K\,\|\lambda(f)\|$$

  for every $f \in C_c(\mathbb{F}_2)$. This shows that φ gives rise to a *multiplier* of $C_r^*(\mathbb{F}_2)$.

Our work started from the desire to highlight the existence of a somewhat hidden track relating these results and related developments in noncommutative harmonic analysis to more traditional issues about the classical theory of Fourier series.

In Section 1 we collect some background material on classical Fourier series, group theory and the various operator algebras associated to (discrete) groups. The presentation covers more topics than strictly needed, with the purpose of providing a reference and a source of inspiration also for future works. In Sections 2 and 3, more biased towards our own contributions [4–8], we present some results illustrating various aspects of Fourier theory in (possibly twisted) discrete reduced group C^*-algebras and reduced C^*-crossed products.

Due to space and time limitations, we have omitted a discussion of a number of different themes, such as noncommutative L^p-spaces, groupoids and quantum groups, where a combination of Fourier theory and C^*-algebras also plays a nontrivial role. The interested reader may for instance consult [9–22] for a small sample.

1. Background

1.1. Classical Fourier series

The classical theory of Fourier series deals with periodic functions on the real line $\mathbb{R}$, i.e., with functions on the torus group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, usually identified with the interval $[-\pi, \pi)$.

Let $f : \mathbb{T} \to \mathbb{C}$ be a function in $L^1(\mathbb{T})$. For every $n \in \mathbb{Z}$, the *n-Fourier coefficient* of f is defined by

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t.$$

It is customary to use the notation $\widehat{f}(n)$ for c_n. The (formal) *Fourier series* of f at $t \in \mathbb{T}$ is given by

$$S[f](t) = \sum_{n\in\mathbb{Z}} c_n e^{int}$$

and, for $N \in \mathbb{N}$, its N-partial sum at $t \in \mathbb{T}$ is defined by

$$S_N[f](t) = \sum_{n=-N}^{N} c_n e^{int}.$$

The original idea of J.B. Fourier of replacing a function f with its Fourier series dates back to 1807 and was aimed at solving a problem of heat diffusion in a metal plate. Nowadays, some basic Fourier analysis, e.g. the Fourier transform $f \mapsto \widehat{f}$ provides a unitary operator from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$ and the L^2-convergence of $S_N[f]$ towards f for every f in $L^2(\mathbb{T})$, is a part of the tool box of many science students. Convergence of Fourier series in other ways gives rise to more delicate problems. Among the impressive body of results, we mention a few highlights.

In 1873, P. du Bois-Reymond showed the existence of a continuous function for which convergence fails at some point. In 1923, A. Kolmogorov produced an example of a function in $L^1(\mathbb{T})$ (but not in $L^2(\mathbb{T})$) having a Fourier series that was divergent almost everywhere (a.e.). He even showed in 1926 that the Fourier series can diverge everywhere. In particular, N. Lusin had asked in 1920 whether the Fourier series of any continuous function converged a.e. This problem was not answered before 1966, when L. Carleson indeed showed that the Fourier series of any function in $L^2(\mathbb{T})$ converges a.e. This result was soon extended by R. Hunt to $L^p(\mathbb{T})$, for any $p > 1$.

If $f \in C(\mathbb{T})$, then $S_N[f]$ converges uniformly (to f) as $N \to \infty$ whenever $S[f]$ is absolutely convergent, that is, whenever $\widehat{f} \in \ell^1(\mathbb{Z})$. This happens for example when f belongs to $C^1(\mathbb{T})$, and the speed of convergence is then known to increase with the smoothness of f, this being reflected in the decay rate of the Fourier coefficients. To the best of our knowledge, no precise characterization of those continuous functions having a uniformly convergent Fourier series is presently available.

Building upon the work of Abel, Cesáro, Poisson, Fejér, and others, there are some well-known procedures, often referred to as summation processes, to enforce uniform convergence by modifying the expression of the Fourier series: for a