Journal of Geometry and Physics 105 (2016) 66–74

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Stability of Lie groupoid C^*-algebras[☆]

Claire Debord[a,*], Georges Skandalis[b]

[a] Laboratoire de Mathématiques, UMR 6620 - CNRS, Université Blaise Pascal, Campus des Cézeaux, BP 80026, F-63171 Aubière cedex, France
[b] Université Paris Diderot, Sorbonne Paris Cité, Sorbonne Universités, UPMC Paris 06, CNRS, IMJ-PRG, UFR de Mathématiques, CP 7012 - Bâtiment Sophie Germain 5 rue Thomas Mann, 75205 Paris CEDEX 13, France

ARTICLE INFO

Article history:
Received 4 March 2015
Accepted 5 March 2016
Available online 18 March 2016

MSC:
primary 58H05
secondary 46L89
58J22

Keywords:
Lie groupoids
Stable C^*-algebras
Singular foliations

ABSTRACT

In this paper we generalize a theorem of M. Hilsum and G. Skandalis stating that the C^*-algebra of any foliation of nonzero dimension is stable. Precisely, we show that the C^*-algebra of a Lie groupoid is stable whenever the groupoid has no orbit of dimension zero. We also prove an analogous theorem for singular foliations for which the holonomy groupoid as defined by I. Androulidakis and G. Skandalis is not Lie in general.

1. Introduction: Statement of the theorem and the steps of the proof

The aim of this paper is to generalize Theorem 1 of [1] stating that the C^*-algebra of any foliation (of nonzero dimension!) is stable.

Theorem 1. *Let G be a Lie groupoid with σ-compact $G^{(0)}$. Assume that at every $x \in G^{(0)}$ the anchor $\natural_x : \mathfrak{g}_x \to T_xG^{(0)}$ is nonzero. Then $C^*(G)$ is stable.*

In other words, $C^*(G)$ is stable whenever G has no orbit of dimension 0. We refer to [2] for the general definition of groupoid C^*-algebras.

The converse is also true if G is s-connected. Indeed, if G is s-connected and the anchor at x is the zero map, then the orbit of x is reduced to x. Therefore $C^*(G)$ has a character: the trivial representation of the group G_x^x.

Since the reduced C^*-algebra $C_r^*(G)$ of G is a quotient of $C^*(G)$, it follows that it is also stable when G has no orbit of dimension 0.

Here however, the converse may fail for the reduced C^*-algebra: the reduced C^*-algebra of the group $PSL_2(\mathbb{R})$ is stable!

Our proof is not very different from the one of [1] and based on Kasparov's stabilization theorem [3]. Note that, unlike in [1], we do not assume the space $G^{(0)}$ to be compact—but this is actually a rather minor point.

The proof is as follows.

☆ The authors were partially supported by ANR-14-CE25-0012-01.
* Corresponding author.
E-mail addresses: claire.debord@math.univ-bpclermont.fr (C. Debord), skandalis@math.univ-paris-diderot.fr (G. Skandalis).

http://dx.doi.org/10.1016/j.geomphys.2016.03.011

1. Let $x \in G^{(0)}$. There is a section Y of the algebroid $\mathfrak{A}(G)$ whose image under the anchor is a vector field X satisfying $X(x) \neq 0$. Taking a local exponentiation of X we obtain a relatively compact open neighborhood W diffeomorphic to $U \times \mathbb{R}$ where X is proportional to the vector field along the $\mathbb{R}$ lines $\{u\} \times \mathbb{R}$.
   This step will be clarified in Section 2.1
   We thus choose a locally finite cover (W_n) by relatively compact open subsets and diffeomorphisms $f_n : U_n \times \mathbb{R} \to W_n$ such that $W'_n = f_n(U'_n \times \mathbb{R})$ cover $G^{(0)}$ with U'_n relatively compact in U_n. Let $p_n : W_n \to U_n$ be the composition of f_n^{-1} with the projection $U_n \times \mathbb{R} \to U_n$.
2. One may then construct a locally finite family of open subsets V_j of $G^{(0)}$ such that:
   - Every V_j sits in a $W_{n(j)}$ and its intersection with each line $f_{n(j)}(u \times \mathbb{R})$ is an (open) interval. More precisely, $f_{n(j)}(V_j)$ is of the form $\{(x, t) \in U_{n(j)} \times \mathbb{R};\ \varphi_j^-(x) < t < \varphi_j^+(x)\}$ where $\varphi_j^-, \varphi_j^+ : U_{n(j)} \to \mathbb{R}$ are smooth and $\varphi_j^+ - \varphi_j^-$ is nonnegative with compact support.
   - The $\overline{V_j}$ are pairwise disjoint and locally finite: every compact subset of M meets only finitely many $\overline{V_j}$'s.
   - For every n, the $p_n(V_j \cap W_n)$ cover U'_n: we have $U'_n \subset \bigcup_{j;\ n(j)=n} p_n(V_j \cap W_n)$.

   The details of the constructions of the V_j's are given in Section 2.2.
3. Let then q_j be the characteristic function of V_j. We prove that q_j is a multiplier of $C^*(G)$. By local finiteness, the characteristic function $q = \sum q_j$ of $V = \bigcup V_j$ is also a multiplier of $C^*(G)$.
   See section Section 3.3.
4. We show that $qC^*(G)$ is a full Hilbert submodule of $C^*(G)$ (see Corollary 9—Section 3.2).
5. Considering a natural diffeomorphism $V_j \simeq p_n(V_j) \times]0, 1[$, it follows that the Hilbert $C^*(G)$-modules $q_jC^*(G)$ and $qC^*(G)$ are stable.
6. Using Kasparov's stabilization Theorem [3], it follows that $C^*(G)$ is stable.
   This follows from Corollary 6—see Section 3.4.

In Section 4, we prove an analogous theorem for singular foliations in the sense of [4]. We prove:

Theorem 2. *The C^*-algebra of a singular foliation (as defined in* [4]*) which has no leaves reduced to a single point is stable.*

The main steps of the proof are the same as for Theorem 1. Using vector fields along the foliation, we construct the same small open subsets V_j. Note that, in proving that the characteristic functions of these V_j are multipliers of $C^*(M, \mathcal{F})$, we chose to take a somewhat different path in order to shed a new light to it. This led us to construct groupoid homomorphisms between singular foliation groupoids (Section 4.2). Of course, we could have used the same kind of proof as for the Lie groupoid case.

2. Geometric constructions

2.1. Nonzero vector fields in the algebroid

Let M be a smooth (open) manifold, x a point of M, and let $X \in \mathfrak{X}(M)$ be a smooth vector field with compact support on M such that $X(x) \neq 0$. Denote by $\Psi_X = (\Psi_X^t)_{t\in\mathbb{R}}$ the flow of X. One can find a codimension one submanifold U of M and a neighborhood I of 0 in $\mathbb{R}$ such that the restriction of Ψ_X to $U \times I$ is a diffeomorphism onto an open (tubular) neighborhood W of U in M. In other words, U is a codimension one submanifold of M which contains x and which is transverse to the integral curves of X.

If G is a Lie groupoid and Y a section with compact support of its Lie algebroid such that the vector field $X := \natural(Y)$ does not vanish on a point $x \in G^{(0)}$, let Z_Y be the associated right invariant vector field on G and Ψ_{Z_Y} its flow. We have $r \circ \Psi_{Z_Y}^t = \Psi^t \circ r$.

Applying the construction above, one finds a codimension one submanifold U of $G^{(0)}$ and a neighborhood I of 0 in $\mathbb{R}$ such that the composition map

$$U \times I \xrightarrow{\Psi_{Z_Y}} G \xrightarrow{r} G^{(0)}$$

is precisely the restriction of the flow Ψ_X of X to $U \times I$ and thus a diffeomorphism onto an open neighborhood W of x in $G^{(0)}$. Note that $s \circ \Psi_{Z_Y}$ is the projection $U \times I \to U$.

Now the following maps are diffeomorphisms:

$$\begin{array}{ccc} G^U \times I & \to & G^W \\ (\gamma, t) & \mapsto & \Psi_{Z_Y}(r(\gamma), t)\gamma \end{array} \quad \text{and} \quad \begin{array}{ccc} G_U^U \times I \times I & \to & G_W^W \\ (\gamma, t, \lambda) & \mapsto & \Psi_{Z_Y}(r(\gamma), t)\gamma\Psi_{Z_Y}(s(\gamma), \lambda)^{-1}. \end{array}$$

2.2. Construction of the family $\boldsymbol{V_j}$

In this section, we explain the construction of the V_j's.

The construction above yields a locally finite cover (W_n) by relatively compact open subsets and diffeomorphisms $f_n : U_n \times \mathbb{R} \to W_n$ such that $W'_n = f_n(U'_n \times \mathbb{R})$ cover $G^{(0)}$ with U'_n relatively compact in U_n. We will often identify W_n and $U_n \times \mathbb{R}$ under f_n.

Let $p_n : W_n \to U_n$ be the composition of f_n^{-1} with the projection $U_n \times \mathbb{R} \to U_n$. As (W_n) is locally finite and $G^{(0)}$ is σ-compact, the set of indices is countable; we identify it with $\mathbb{N}$.