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Dixmier traces and non-commutative analysis

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ABSTRACT

In the present paper we review recent advances in the theory of Dixmier traces and aspects of their application to noncommutative analysis and geometry. We describe J. Dixmier's original construction of singular traces together with recent revisions of his ideas. We pay particular attention to subclasses of Dixmier traces related to exponentiation invariant extended limits and notions of measurability due to A. Connes. We discuss in detail the applications of Dixmier traces to the study of spectral properties of pseudo-differential operators and a very recent application of Dixmier traces in the study the Fréchet differentiability of Haagerup's L_p norm.

1. An introduction to singular traces

Let $B(H)$ be the algebra of all bounded linear operators on a separable Hilbert space H. By $\{\lambda(k, A)\}_{k=0}^{\infty}$ we denote an eigenvalue sequence of a compact operator $A \in B(H)$ counted with multiplicities and arranged in a non-increasing order. Consider the trace-class ideal $\mathcal{L}_1$ in $B(H)$ consisting of all compact operators $A \in B(H)$ such that

$$\sum_{k=0}^{\infty} |\lambda(k, A)| < \infty.$$

The canonical trace on $\mathcal{L}_1$ can be defined by the following formula:

$$\mathrm{Tr}(A) = \sum_{k=0}^{\infty} \lambda(k, A), \quad A \in \mathcal{L}_1.$$

This trace is normal in the sense that $\mathrm{Tr}(\sup_\alpha A_\alpha) = \sup_\alpha \mathrm{Tr}(A_\alpha)$ holds for every bounded increasing directed family of positive operators $\{A_\alpha\}_\alpha \in B(H)$. It is well-known that the normal trace is essentially unique, that is, every normal trace is proportional to the canonical trace. Therefore the following question arises naturally. Are there any other traces besides the canonical trace? Do non-normal traces exist?

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This question was first stated and resolved in the affirmative by J. Dixmier [1]. For positive compact operators $A \in B(H)$ satisfying the following condition:

$$\sup_{n\geq 1} \frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k, A) < \infty \tag{1}$$

he defined a weight[1]

$$t(A) := \lim \frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k, A), \quad A \geq 0. \tag{2}$$

There are two issues with the definition of the weight t in (2). First of all, the condition (1) does not guarantee the convergence of the sequence $\left\{\frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k, A)\right\}_{n\geq 1}$. So, instead of the ordinary limit, a "generalised" limiting procedure should be considered. Secondly, to extend this weight by linearity to a trace one needs to prove that it is positive homogeneous, which is straightforward, and additive, which is difficult, on the positive cone.

It turned out that both of these issues can be resolved by a suitable choice of the linear form *lim*. J. Dixmier chose lim to be a restriction to sequences of a linear form invariant under the group of affine transformations $t \mapsto at + b$ on $\mathbb{R}$, that is, a functional on $\mathbb{R}$ invariant under dilations and translations.

We formulate Dixmier's result in a simpler form due to A. Connes [2]:

Theorem 1.1. *If ω is a state on the space l_∞ of bounded sequences, such that*

(i) *ω vanishes on finitely supported sequences,*

(ii) *$\omega(x_0, x_1, \ldots) = \omega(x_0, x_0, x_1, x_1, \ldots)$ for every $x \in l_\infty$, then the weight*

$$\mathrm{Tr}_\omega(A) := \omega\left(\frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k, A)\right) \tag{3}$$

is positive homogeneous and additive on the cone of all positive compact operators $A \in B(H)$ satisfying (1).

Let Tr_ω also denote the extension by linearity of the weight defined by (3) to the whole space of operators satisfying (1). We list the properties of this functional:

1. Since eigenvalues of a compact operator are a unitary invariant, it follows that the functional Tr_ω is unitary invariant, that is, Tr_ω is a trace on $B(H)$;

2. The trace Tr_ω is non-trivial since $\mathrm{Tr}_\omega(\mathrm{diag}\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}) = 1$;

3. If A is a finite rank operator then $\mathrm{Tr}_\omega(A) = 0$ due to the properties of the state ω. Moreover, $\mathrm{Tr}_\omega(A) = 0$ for every $A \in \mathcal{L}_1$.

Theorem 1.1 provides an example of a non-trivial and non-normal trace Tr_ω. Such traces are termed Dixmier traces in acknowledgement of J. Dixmier's contribution.

2. Dixmier traces and measurability

In his construction J. Dixmier first fixed a state on the space of bounded functions and then employed the restriction of this state to the space of bounded sequences. It turned out to be convenient to do the opposite; to embed the space l_∞ of all bounded sequences into the Lebesgue space $L_\infty(0, \infty)$ and then work with a state on the space of all essentially bounded functions (see e.g. [3–5]).

In the present section we outline the latter approach to Dixmier traces and show that it is equivalent to the original construction of J. Dixmier [1]. Our exposition complements that of [6,7].

2.1. Extended limits

We will denote by $L_\infty(0, \infty)$ and by $L_\infty(\mathbb{R})$ the spaces of all (classes of) real-valued essentially bounded Lebesgue measurable functions on $(0, \infty)$ and $\mathbb{R}$, respectively, equipped with the norm

$$\|x\|_{L_\infty} := \mathrm{esssup}|x(t)|,$$

where essential supremum is taken over all $t > 0$ and $t \in \mathbb{R}$, respectively.

[1] In fact, Dixmier worked in the more general setting of the Lorentz ideal $\mathcal{M}_\psi$ of compact operators with partial sums of eigenvalues that diverge as a given concave function $\psi : (0, \infty) \to (0, \infty)$ (see Section 2.2 for definitions).