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Some C^*-algebras which are coronas of non-C^*-Banach algebras

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A B S T R A C T

We present results and motivating problems in the study of commutants of hermitian n-tuples of Hilbert space operators modulo normed ideals. In particular, the C^*-algebras which arise in this context as coronas of non-C^*-Banach algebras, the connections with normed ideal perturbations of operators, the hyponormal operators and the bidual Banach algebras one encounters are discussed.

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1. Introduction

Starting with a Banach algebra, which is not a C^*-algebra, taking the multiplier algebra and then the quotient by the initial algebra, that is, passing to the corona of the Banach algebra, it can happen that the corona is a C^*-algebra. We recently encountered such C^*-algebras, that are naturally C^*-subalgebras of the Calkin algebra, in the study of the commutants modulo certain normed ideals of n-tuples of Hilbert space operators [1–3].

In [4–7] we had developed a machinery for the study of normed ideal perturbations of n-tuples of operators, based on an adaptation of our non-commutative Weyl–von Neumann type theorem [8] to normed ideals which may be smaller than the ideal of compact operators. Also related to this in [9] we had raised the question whether there may be an analogue of the Brown–Douglas–Fillmore theorem [10] for operators with trace-class self-commutator where the Pincus g-function [11] plays the role of the index and the unitary equivalence is modulo the Hilbert–Schmidt class. Recently we realized that these kinds of questions involve the K-theory of the commutant modulo normed ideals, like the Hilbert–Schmidt class, of certain n-tuples of operators and began looking at this more generally.

Loosely, there are two further sources for our renewed interest in these problems, we would like to mention. On one hand operators with trace-class self-commutator, or equivalently pairs of hermitian operators with trace-class commutator, appear in many current questions. For instance, this is the case with rank-one commutators and the analogue of the Fisher

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information in free probability [12]. On the other hand there is the work of Alain Connes [13] on the spectral characterization of smooth compact manifolds (that is their characterization as non-commutative manifolds) where our normed ideal perturbation results found technical uses.

Besides their natural role in normed ideal perturbations of operators, the algebras considered here may also be a testing ground of the developments of KK-theory beyond C^*-algebra [14]. Note that the K-theory of these algebras is different, probably much richer than that of the related C^*-algebras of C^*-algebraic origin.

The present paper is an expanded version of the lecture we gave at the NGA Conference at Villa Mondragone in June 2014, providing in particular more detailed background material.

2. Normed ideal perturbations machinery background

2.1. Preliminaries on normed ideals

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $\mathcal{R}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$ denote the finite rank and respectively the compact and bounded operators on $\mathcal{H}$. The *normed ideals* $(\mathcal{J}, |\ |_{\mathcal{J}})$, an abbreviation for *symmetrically normed ideals* (see [15,16] for details on definitions and properties) are ideals $\mathcal{J}$ in $\mathcal{B}(\mathcal{H})$ endowed with the norm $|\ |_{\mathcal{J}}$ with respect to which they are Banach spaces and $|AXB|_{\mathcal{J}} \le \|A\||X|_{\mathcal{J}}\|B\|$ if $A, B \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{J}$. We assume $\mathcal{R}(\mathcal{H}) \subset \mathcal{J} \subset \mathcal{K}(\mathcal{H})$ and $\|X\| = |X|_{\mathcal{J}}$ if rank $X = 1$.

If $\Phi(s_1 \ge s_2 \ge \cdots \ge 0)$ is a norming function (see [15,16] for the definition) and $X \in \mathcal{K}(\mathcal{H})$ and $s_1 \ge s_2 \ge \cdots$ are the eigenvalues of $(X^*X)^{1/2}$ let $|X|_\Phi = \Phi(s_1 \ge s_2 \ge \cdots)$ and let $\mathfrak{S}_\Phi = \{X \in \mathcal{K}(\mathcal{H}) \mid |X|_\Phi < \infty\}$ and $\mathfrak{S}_\Phi^{(0)}$ be the closure of $\mathcal{R}(\mathcal{H})$ in $\mathfrak{S}_\Phi$, with respect to the norm $|\ |_\Phi$. Then $(\mathfrak{S}_\Phi^{(0)}, |\ |_\Phi)$ and $(\mathfrak{S}_\Phi, |\ |_\Phi)$ are normed ideals and for every normed ideal $(\mathcal{J}, |\ |_{\mathcal{J}})$ there is a norming function Φ so that $\mathfrak{S}_\Phi^{(0)} \subset \mathcal{J} \subset \mathfrak{S}_\Phi$ and $|\ |_{\mathcal{J}} = |\ |_\Phi$. The norming function Φ is called *mono-norming* if $\mathfrak{S}_\Phi^{(0)} = \mathfrak{S}_\Phi$. The Schatten–von Neumann p-class $\mathcal{C}_p$ is the normed ideal corresponding to the norming function $\Phi_p(s_1 \ge s_2 \ge \cdots) = \left(\sum_j s_j^p\right)^{1/p}$, $1 \le p < \infty$, which is mono-norming. By $\mathcal{C}_p^-$ we shall denote the normed ideal for the mono-norming function $\Phi_p^-(s_1 \ge s_2 \ge \cdots) = \sum_j s_j j^{-1+1/p}$, $1 \le p \le \infty$. On the Lorentz-scale with two indices $\mathcal{C}_p^-$ is $\mathcal{C}_{(p,1)}$ (see [15,16]). We have $\mathcal{C}_p^- \subset \mathcal{C}_p$, if $p' < p$ we have $\mathcal{C}_{p'} \subset \mathcal{C}_p^-$ and $\mathcal{C}_1^- = \mathcal{C}_1$.

For more on normed ideals we refer the reader to the books [15,16].

2.2. The invariant $k_{\mathcal{J}}(\tau)$

Definition. Let $(\mathcal{J}, |\ |_{\mathcal{J}})$ be a normed ideal and $\tau = (T_1, \ldots, T_n) \in (\mathcal{B}(\mathcal{H}))^n$. We define

$$k_{\mathcal{J}}(\tau) = \liminf_{A \in \mathcal{R}_1^+(\mathcal{H})} |[A, \tau]|_{\mathcal{J}}$$

where $\mathcal{R}_1^+(\mathcal{H}) = \{A \in \mathcal{R}(\mathcal{H}) \mid 0 \le A \le I\}$, the liminf being with respect to the natural order on $\mathcal{R}_1^+(\mathcal{H})$, $[A, \tau] = ([A, T_1], \ldots, [A, T_n])$ and $|[A, \tau]|_{\mathcal{J}} = \max_{1 \le j \le n} |[A, T_j]|_{\mathcal{J}}$. If Φ is a norming function such that $|\ |_{\mathcal{J}} = |\ |_\Phi$, $k_{\mathcal{J}}(\tau)$ will also be denoted $k_\Phi(\tau)$ (then $\mathcal{J}$ could be any normed ideal so that $\mathfrak{S}_\Phi^{(0)} \subset \mathcal{J} \subset \mathfrak{S}_\Phi$).

We have $k_{\mathcal{J}}(\tau) = 0$ iff there are $A_n \in \mathcal{R}_1^+(\mathcal{H})$ so that $A_n \uparrow I$ and $|[A_n, \tau]|_{\mathcal{J}} \to 0$ as $n \to \infty$. The number $k_{\mathcal{J}}(\tau)$ measures the obstruction to the existence of a sequence of A_n, $n \in \mathbb{N}$ with the above properties. Note that $k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\tau^*) = k_{\mathcal{J}}(\tau, \tau^*)$ and $k_{\mathcal{J}}(\tau) \asymp k_{\mathcal{J}}(\mathrm{Re}\,\tau, \mathrm{Im}\,\tau)$, so that the study of $k_{\mathcal{J}}(\tau)$ boils down to the case of n-tuples of hermitian operators. Note also that if τ and τ' are congruent mod $\mathcal{J}$ then $k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\tau')$. To give the reader some feeling about the properties of $k_{\mathcal{J}}(\tau)$ we shall mention some results in what follows but there is much more, for which we refer the reader to [4–6,17], the survey [7] and further references therein.

If $\tau = \tau^*$ is an n-tuple of commuting hermitian operators and k_p, k_p^- stand for $k_{\mathcal{C}_p}$, $k_{\mathcal{C}_p^-}$, we have $k_p(\tau) = 0$ if $p \ge n \ge 2$ and $k_n^-(\tau) > 0$ iff τ has non-empty n-dimensional Lebesgue absolutely continuous spectrum. The ideal $\mathcal{C}_\infty^-$ is the largest ideal for which $k_{\mathcal{J}}(\tau)$ may be non-zero. We have that $k_\infty^-(\tau)$ is always $< \infty$ and if $\mathcal{J} \supset \mathcal{C}_\infty^-$, $\mathcal{J} \ne \mathcal{C}_\infty^-$ then $k_{\mathcal{J}}(\tau) = 0$. If τ is a n-tuple of Cuntz isometries $n \ge 2$ then $k_\infty^-(\tau) > 0$. Actually if we consider for instance $k_p^-(\tau)$ as a function of $p \in [1, \infty]$ for any τ, then there is some $p_0 \in [1, \infty]$, depending on τ, so that $k_p^-(\tau) = \infty$ if $p \in [1, p_0)$ and $k_p^-(\tau) = 0$ if $p \in (p_0, \infty]$.

The non-vanishing of $k_\Phi(\tau)$ is equivalent to a condition involving traces of commutators and the dual normed ideal $(\mathfrak{S}_\Phi^{(0)})^{\mathrm{dual}} = \mathfrak{S}_{\Phi^d}$.

Fact 2.1 (*[6]*). If $\tau = \tau^*$, then $k_\Phi(\tau) > 0$ iff there exist $Y_j = Y_j^* \in (\mathfrak{S}_\Phi^{(0)})^{\mathrm{dual}}$ so that $\sum_j i[T_j, Y_j] = Z$ with $\mathrm{Tr}\, Z \in (0, \infty]$.

2.3. The adapted non-commutative Weyl–von Neumann type theorem

Fact 2.2 (*[4]*). Let A be a separable unital C^*-algebra and let $\omega = (X_k)_{k \in \mathbb{N}} \subset A$, $X_j = X_j^*$ be a generating set. Let $\rho_j : A \to \mathcal{B}(\mathcal{H})$ $j = 1, 2$ be faithful unital $*$-representations such that $\rho_j(A) \cap \mathcal{K}(\mathcal{H}) = 0$ and assume $k_\Phi((\rho_j(X_k))_{1 \le k \le n}) = 0$for $j = 1, 2$ and

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