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Spectral stability analysis for special solutions of second order in time PDEs: The higher dimensional case

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HIGHLIGHTS

- General theory for the stability of standing waves of second-order in time PDEs.
- Obtains results on special solutions of the Klein–Gordon equation, the KGZ-system, etc.
- Results are applicable to multi-dimensional traveling waves and standing–traveling waves.

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ABSTRACT

We develop a general theory to treat the linear stability of certain special solutions of second order in time evolutionary PDE. We apply these results to standing waves of the following problems: the Klein–Gordon equation, for which we consider both ground states and certain excited states, the Klein–Gordon–Zakharov system and the beam equation. We also discuss applications to excited states for the Klein–Gordon model as well as multidimensional traveling waves (not necessarily homoclinic to zero) for general second order equations of this type. In all cases, our abstract results provide a complete characterization of the linear stability of such solutions.

1. Introduction

In this article, we consider second order in time evolutionary equations/systems in the form

$$u_{tt} + \mathcal{L}u - f(|u|^2)u = 0$$
$$(t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^d \text{ or } (t,x) \in \mathbf{R}^1_+ \times [-L,L]^d, \quad (1)$$

where the nonlinearity $f : \mathbf{R}^1 \to \mathbf{R}^1$ and the (unbounded) self-adjoint linear differential operator $\mathcal{L}$ are to be made precise in each concrete example.

We will be interested in the linear stability of various special solutions of nonlinear PDEs. In order to focus the discussion, we start with the most natural example, which fits our framework — the standing wave solutions of (1). These objects have been studied extensively in the last thirty years and many methods have been developed to study their stability properties. We would like to use them as a starting example, in order to motivate our approach and the abstract results that will address these issues.

Going back to the standing wave solutions, these are solutions in the form $u(t,x) = e^{i\omega t}\varphi_\omega(x)$, where $\omega \in \mathbf{R}^1$ and φ_ω is real-valued. Such solutions satisfy the stationary equation

$$\mathcal{L}\varphi - \omega^2\varphi - f(\varphi^2)\varphi = 0. \quad (2)$$

In order to ease into the notion of linear stability, which will be the main focus, let us consider the linearization of Eq. (1). To that end, let $u = e^{i\omega t}(\varphi_\omega(x) + v(t,x))$ and plug it into (1). This is of course still a nonlinear equation for v. Assuming that v is small, it is reasonable to ignore all the terms in the form $O(v^2)$. We arrive at the following *linear* equation for v:

$$v_{tt} + 2i\omega v_t - \omega^2 v + \mathcal{L}v - f(\varphi^2)v - 2\varphi^2 f'(\varphi^2)\Re v = 0. \quad (3)$$

Separating the real and imaginary parts, with the assignment $\mathbf{v} = (\Re v, \Im v)$, yields the following system for $\mathbf{v}$:

$$\mathbf{v}_{tt} + 2\omega J\mathbf{v}_t + \mathcal{H}\mathbf{v} = 0, \quad (4)$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},$$

$$L_+ = \mathcal{L} - \omega^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2)$$
$$L_- = \mathcal{L} - \omega^2 - f(\varphi^2).$$

Note that if the function f is increasing, the self-adjoint operators satisfy $L_- \geq L_+$.

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We would like to point out that the standing wave solutions are by no means the only example that fits our theory. As we shall see below, our results are applicable to multi-dimensional traveling waves as well as standing–traveling waves. Additional applications include the recent work by Stanislavova [1] where the stability of subsonic traveling waves for the Benney–Luke model is completely characterized.

In order to give a definition of linear stability, we assume that the linear system (4) has global solutions for all sufficiently smooth and decaying data. This is of course equivalent to saying that the operator

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & 1 \\ -\mathcal{H} & -2\omega J \end{pmatrix}$$

generates a C_0 semi-group on appropriate spaces, but this is sometimes hard to verify in concrete examples. In any case, under this assumption, we say that the standing wave $e^{i\omega t}\varphi_\omega$ is *linearly stable*, if the solution to the linear system (4) satisfies $\lim_{t\to\infty} e^{-\delta t}\|\mathbf{v}(t)\| = 0$ for any $\delta > 0$ and for a dense set of appropriate initial data.

Similarly, we say that the system is *spectrally stable*, if the spectrum of $\tilde{H}$ lies in the closed left half-space. That is $\sigma(\tilde{\mathcal{H}}) \subseteq \{z : \Re z \le 0\}$. Note that under the standard assumption that $\tilde{\mathcal{H}}$ generates a C_0 semi-group, linear stability implies spectral stability, but not vice versa. Under some natural extra assumptions however (which guarantee the validity of the so-called spectral mapping theorem), the spectral stability is indeed equivalent to linear stability. We will not explore this connection any further, but the interested reader can consult the book [2], the excellent survey paper [3] as well as [4,5].

One also has the related notion of nonlinear (orbital) stability. Basically, this means that if one starts close to the standing wave, then the solution will stay close to the wave, modulo the invariance of the system under consideration. The notion of asymptotic stability is the strongest of all and it requires the difference between the two close solutions (modulo the invariance) to go to zero as time goes to infinity. We will not pursue these issues here, except to mention that establishing linearized stability is a prerequisite for asymptotic stability results, and thus, the results in this paper should be viewed as an important step toward accomplishing such a goal. For examples of asymptotic and orbital stability results see [6–8].

1.1. Examples

We consider the following models — the Klein–Gordon equation, the Klein–Gordon–Zakharov system and the beam equation in the whole space contexts, although the methods developed herein will be certainly useful for other examples and/or periodic domains. Also, we mainly consider standing wave solutions, although toward the end of the discussion, we offer some ideas on how to obtain stability/instability results for multidimensional traveling waves as well; see Section 2.5.

All of these models have been the subject of an intensive investigation in the last thirty years, with the majority of the results concerning orbital stability/instability. This was partly due to the versatility of the general theory, developed by Grillakis–Shatah–Strauss for such equations/systems. We provide more specific references to these studies after our theorems, which once again concern *the linear stability* of their special solutions.

We begin with some basic setup, which has dual purpose: on the one hand, it will motivate our approach to the problem at hand, and on the other, it will set the stage for the proofs in the subsequent sections. We start with the Klein–Gordon model.

1.1.1. Klein–Gordon equation: ground states

Consider

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0. \tag{5}$$

This clearly fits the profile (1), where the operator $\mathcal{L} := -\Delta + 1$. It is well-known that in this case, the corresponding operator $\tilde{\mathcal{H}}$ generates a C_0 semigroup; see [9]. Let us consider some general properties of the operators $\mathcal{H}, L_\pm$, depending on the type of solutions φ_ω that one encounters. Observe that, if we consider only decaying solutions of (2), we can conclude that $\sigma_{a.c.}(L_\pm) \subset [1-\omega^2, \infty)$ by Weyl's theorem. Note that by (2), $L_-[\varphi] = 0$. Moreover, if φ does not change sign (say, we take it to be positive), it follows by Sturm–Liouville's theory that $L_- \ge 0$ and 0 is a simple eigenvalue. That is $\sigma(L_-) \subset [0, \infty)$ and $L_-|_{\{\varphi\}^\perp} \ge \kappa^2 > 0$.

In addition, differentiating (2) with respect to the spatial variables produces the identity $L_+[\nabla_x \varphi] = 0$, whence $Ker[L_+]$ is at least d dimensional, with eigenfunctions $\frac{\partial\varphi}{\partial x_j} : j = 1, \dots, d$. Usually, $Ker[L_+] = span[\frac{\partial\varphi}{\partial x_j} : j = 1, \dots, d]$, but this is by no means automatic. Note also that

$$\langle L_+[\varphi], \varphi\rangle = -(p-1)\int \varphi^{p+1}(x)dx < 0,$$

thus guaranteeing the presence of a negative point spectrum for L_+. In the seminal papers by Shatah, [10] and Weinstein, [11], most of the spectral properties for the operators $L_\pm$ were established. The full and complete analysis of the spectral properties of $L_\pm$ was subsequently given by Kwong in [12]. We also recommend the excellent paper [13] for a more contemporary approach to these facts.

To summarize the known results in the case of power nonlinearities, for $p \in (1, p_{\max})$,

$$p_{\max} = \begin{cases} 1 + \dfrac{4}{d-2} & d \ge 3 \\ \infty & d = 1, 2 \end{cases}$$

we have $L_- \ge 0$, $L_-[\varphi] = 0$, $L_-|_{\{\varphi\}^\perp} \ge \kappa^2 > 0$, while $L_+ : Ker[L_+] = span[\frac{\partial\varphi}{\partial x_j} : j = 1, \dots, d]$, with single simple negative eigenvalue, $L_+[\phi] = -\sigma_0^2\phi$ and $L_+|_{\{\phi, \nabla\varphi\}^\perp} \ge \kappa^2 > 0$.

We now turn our attention to the problem for excited states of the Klein–Gordon model.

1.1.2. Klein–Gordon: excited states (vortices) in two dimensions

Besides the ground states solutions, whose properties were described in Section 1.1.1 above, there are numerous other "excited" solutions of (5). For example, P.L. Lions, [14] has constructed stationary solutions in even dimensions[2] $d = 2k$ in the form

$$\phi(r_1, \dots, r_k)e^{i(m_1\theta_1 + \cdots m_k\theta_k)}, \quad m \in \mathbf{Z}^k$$

where $(r_j, \theta_j), j = 1, \dots, k$ are the polar variables corresponding to (x_{2j-1}, x_{2j}). In the case of two spatial dimensions, this work has been extended by Iaia and Warchal, [15], who have shown that there are infinitely many solutions in the form $\phi_{m,k,p}(r)e^{im\theta}$. More precisely, these satisfy

$$-\phi''(r) - \frac{1}{r}\phi'(r) + \frac{m^2}{r^2}\phi(r) + \phi(r) - |\phi(r)|^{p-1}\phi = 0 \tag{6}$$

where Eq. (6) is supplied by the natural boundary conditions $\lim_{r\to 0+} r^{-m}\phi(r) = 0$, $\lim_{r\to 0+} r^{-m+1}\phi'(r) = m\alpha$ for some $\alpha \ge 0$ and k stands for the number of zeros of $\phi_{m,k,p}(r)$. In a subsequent work, Mizumachi, [16] has shown the uniqueness of the positive solutions of (6) (i.e. for $k = 0$) and in addition, he has shown the orbital stability for $1 < p < 3$ (and instability for $p > 3$) of the standing waves $e^{i(\omega t + m\theta)}\phi_{m,0,p}(r)$, where these are understood as time periodic solutions to the Schrödinger equation and the perturbations are taken to be in the form $e^{i(\omega t + m\theta)}z(r)$. We encourage the reader to consult the excellent paper, [13], where these and other results are reviewed in full detail, including a number of high-precision numerical verifications thereof.

[2] And similarly in odd dimensions, which we do not consider herein.