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Potts models with magnetic field: Arithmetic, geometry, and computation

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ABSTRACT

We give a sheaf theoretic interpretation of Potts models with external magnetic field, in terms of constructible sheaves and their Euler characteristics. We show that the polynomial countability question for the hypersurfaces defined by the vanishing of the partition function is affected by changes in the magnetic field: elementary examples suffice to see non-polynomially countable cases that become polynomially countable after a perturbation of the magnetic field. The same recursive formula for the Grothendieck classes, under edge-doubling operations, holds as in the case without magnetic field, but the closed formulae for specific examples like banana graphs differ in the presence of magnetic field. We give examples of computation of the Euler characteristic with compact support, for the set of real zeros, and find a similar exponential growth with the size of the graph. This can be viewed as a measure of topological and algorithmic complexity. We also consider the computational complexity question for evaluations of the polynomial, and show both tractable and NP-hard examples, using dynamic programming.

1. Introduction

Several combinatorial graph polynomials have physical significance, either as partition functions of statistical mechanical models on graphs (Ising and Potts models), or as the Kirchhoff and Symanzik polynomials that appear in the parametric form of Feynman integrals in perturbative quantum field theory. In both cases, it is interesting to consider various questions related to the properties of these polynomials and of the hypersurfaces they define. For a survey of the quantum field theory case, we refer the reader to [1], and for the case of Potts models, to [2,3] and to the general survey [4].

In this paper, we focus on another such polynomial with physical significance: the **V**-polynomial, which gives the partition function of the Potts models with external magnetic field.

After recalling some general facts about these polynomials, we show in Section 2 that they admit a sheaf theoretic interpretation as the Euler characteristics of a constructible complex $\mathcal{F}^\bullet_\Gamma$ over the graph configuration space $\mathrm{Conf}_\Gamma(X)$ of a smooth projective variety. This addresses a question posed to the second author by Spencer Bloch.

In Section 3, we consider the hypersurfaces defined by the vanishing of the **V**-polynomial, and the question of whether these varieties are polynomially countable, that is, whether the counting of points over finite fields $\mathbb{F}_p$ is a polynomial in p. In the case of quantum field theory, the analogous polynomial countability question has drawn a lot of attention in recent years, in relation to questions on the occurrence of motives and periods in Feynman integrals. Counterexamples to polynomial countability for the Kirchhoff polynomials of quantum field theory are very elusive, and only occur for combinatorially complicated graphs with a large number of edges and loops (see the recently found examples in [5] and [6]). It is much

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simpler to find non-polynomially countable examples in the case of the Potts model partition function, see [3]. As expected, even smaller graphs give rise to non-polynomially countable hypersurfaces in the case of the **V**-polynomial. However, a new phenomenon occurs: polynomial countability depends on the magnetic field, and can occasionally be restored by modifying the magnetic field. We illustrate these phenomena in the simplest examples.

In Section 4 we consider the class in the Grothendieck ring of varieties of the complement of the hypersurface defined by the vanishing of the **V**-polynomial. We show that the same recursive formula for edge-doubling, proved in [2] in the case without magnetic field, continues to hold in this case. However, the presence of magnetic field alters the initial terms of the recursion. We compute the resulting closed form of the class for the case of banana graphs and compare it with the case without magnetic field. As in [2], we then focus on the set of real zeros, and its Euler characteristic with compact support, as a measure of complexity (topological and algorithmic) of the analytic set of real zeros. We provide simple examples where this quantity grows exponentially with the size of the graph.

In Section 5 we consider a different kind of complexity question regarding the **V**-polynomials, namely the computational complexity of evaluating at a point. Using dynamic programming, we show that line and polygon graphs are tractable, while full binary trees, and trees that limit to the line are NP-hard.

1.1. The **V**-polynomial

The correspondence between the Tutte polynomial and the partition function assumes a zero-field Hamiltonian [7], which excludes several important cases, including the presence of an external magnetic field. However, there exists a combinatorial polynomial that is the evaluation of the Potts model with an external field, the **V**-polynomial. In this paper, we will study the algebraic, topological, and computational complexity of the **V**-polynomial.

Let Γ be a finite graph, with edge set $E(\Gamma)$ and vertex set $V(\Gamma)$. A vertex weight on Γ is a function $\omega : V(\Gamma) \to S$, with S a torsion-free abelian semigroup.

We recall from [7] the definition of the **V**-polynomial. It is a polynomial in $\mathbb{Z}[t = (t_e)_{e\in E(\Gamma)}, x = (x_s)_{s\in S}]$, where the t_e are edge variables (edge weights), and the x_s account for the presence of the magnetic field. We view the **V**-polynomial as a map $\mathbf{V}_\Gamma : \mathbb{A}^{\#E(\Gamma)} \times \mathbb{A}^{\#S} \to \mathbb{A}$. For a subset $A \subseteq E(\Gamma)$, we denote by $\Gamma_A \subset \Gamma$ the subgraph of Γ with $V(\Gamma_A) = V(\Gamma)$ and $E(\Gamma_A) = A$. Let $\Gamma_{A,j}$, for $j = 1, \dots, b_0(\Gamma_A)$ be the connected components of Γ_A. Then the **V**-polynomial is defined as

$$\mathbf{V}_\Gamma(t, x) = \sum_{A\subseteq E(\Gamma)} \prod_{j=1}^{b_0(\Gamma_A)} x_{s_j} \prod_{e\in A} t_e, \tag{1.1}$$

where $s_j = \sum_{v\in\Gamma_{A,j}} \omega(v)$ is the sum of the weights attached to all the vertices in the jth component.

The **V**-polynomial is determined recursively by the deletion–contraction relation.

- For an edge $e \in E(\Gamma)$ that is not a looping edge,

$$\mathbf{V}_\Gamma(t, x) = \mathbf{V}_{\Gamma\smallsetminus e}(\hat{t}, x) + t_e \mathbf{V}_{\Gamma/e}(\hat{t}, x), \tag{1.2}$$

  with $\hat{t}$ the vector of edge variables with t_e removed.
- For a looping edge e

$$\mathbf{V}_\Gamma(t, x) = (t_e + 1)\mathbf{V}_{\Gamma\smallsetminus e}(\hat{t}, x). \tag{1.3}$$

- If Γ consists of a set of vertices $V(\Gamma)$ and no edges, $E(\Gamma) = \emptyset$, then

$$\mathbf{V}_\Gamma(t, x) = \prod_{v\in V(\Gamma)} x_{\omega(v)}, \tag{1.4}$$

  where $\omega : V(\Gamma) \to S$ is the vertex weight.

Relations between the **V**-polynomial, the W-polynomial of [8], and the multivariable Tutte polynomial are described in [7].

1.2. The **V**-polynomial and magnetic field

The physical interpretation of the **V**-polynomial as partition function of the Potts model with magnetic field comes from rewriting the Fortuin–Kasteleyn representation (1.1) of the polynomial as the partition function

$$Z_\Gamma = \sum_{A\subseteq E(\Gamma)} \prod_{j=1}^{b_0(\Gamma_A)} X_{M_{c_j}} \prod_{a\in A} (e^{-\beta J_e} - 1),$$

where β is a thermodynamic inverse temperature parameter, the J_e are the nearest-neighbor interaction energies along the edges, and M is the magnetic field vector, with

$$X_{M_{c_j}} = \sum_{v\in V(\Gamma_{A,j})} e^{-\beta M_v},$$

with $\Gamma_{A,j}$ the jth connected component of the graph Γ_A.