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Leibniz algebroids, generalized Bismut connections and Einstein–Hilbert actions

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ABSTRACT

Connection, torsion and curvature are introduced for general (local) Leibniz algebroids. Generalized Bismut connection on $TM \oplus \Lambda^p T^*M$ is an example leading to a scalar curvature of the form $R + H^2$ for a closed $(p+2)$-form H.

1. Introduction

In this short note, we start to develop a general theory of connections, torsions and curvatures for local Leibniz algebroids. Interesting Leibniz algebroids are, for instance, those related to exceptional generalized geometries [1]. We believe that our constructions can be applied to a wide class of closed form Leibniz algebroids, classified in [2]. In all these Leibniz algebroids we have, in addition to the anchor controlling the Leibniz property in the second argument of the Dorfman bracket, also the so called locality operator controlling the behavior of the bracket under the multiplication of its first argument by a function. This locality operator can then be used to define the appropriate notions of torsion and curvature.

Though the general theory is simple and transparent, explicit computations are quite tedious even in the simplest examples. Hence, in this short note we present as an example only results for the simplest Leibniz algebroid on $TM \oplus \Lambda^p T^*M$ equipped with the higher Dorfman bracket and with the corresponding generalized metric (defined by an ordinary Riemannian metric g and by a $(p+1)$-form C). In this example, we generalize the (generalized) Bismut connection from the case $p = 1$, whose significance in the context of generalized geometry was first understood and investigated in [3]. Its properties were highlighted in [4], where its torsion was defined too. The calculations relating such metric connections with skew torsion to the Courant bracket go back to [5,6].

Of course, what we aim for are general definitions that in our example lead to the scalar curvature of the form $R + (dC)^2$.

There is a vast and important literature on supergravity actions from the point of view of (exceptional) generalized geometry and/or double field theory. It is far beyond the scope of this short note to comment on all of these, even to cite

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them. Among these, it seems to us that [7,8] are, at least in some aspects, closest to our point of view and include an excellent overview of the literature.

2. Local Leibniz algebroids

Let us recall the notion of a Leibniz (Loday) algebroid. A Leibniz algebroid is a triple $(E, \rho, \circ)$, where $E \xrightarrow{\pi} M$ is a (smooth) vector bundle, $\rho : E \to TM$ is a vector bundle morphism, called the anchor, and $\circ$ is an $\mathbb{R}$-bilinear bracket on sections $\Gamma(E)$ of E, satisfying the Leibniz rule

$$e \circ (fe') = f(e \circ e') + (\rho(e).f)e' \tag{1}$$

and the Leibniz identity

$$e \circ (e' \circ e'') = (e \circ e') \circ e'' + e' \circ (e \circ e'') \tag{2}$$

for all $e, e', e'' \in \Gamma(E)$ and $f \in C^\infty(M)$.

From the consistency of the Leibniz rule and the Leibniz identity under the replacement $e'' \mapsto fe''$, it follows that $\rho(e \circ e') = [\rho(e), \rho(e')]$. Further, we have a natural "differential" d[1] defined as an $\mathbb{R}$-linear map $\mathrm{d} : C^\infty(M) \to \Gamma(E^*)$

$$\langle \mathrm{d}f, e\rangle = \rho(e).f \tag{3}$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$. Obviously d satisfies the usual Leibniz rule $\mathrm{d}(fg) = \mathrm{d}(f)g + f\mathrm{d}(g)$, for $f, g \in C^\infty(M)$.

Next, one can define a Lie derivative $\mathcal{L}^E$, corresponding to $\circ$. It will define a first order differential operator on the tensor bundle $\mathcal{T}(E)$. It is defined in the lowest orders and extended as a differential to all tensors. On functions, it just the derivative in the direction of $\rho(e)$

$$\mathcal{L}^E_e f = \rho(e).f \tag{4}$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M) = \Gamma(\mathcal{T}_0^{\,0}(E))$. On sections of E, it is the Leibniz bracket $\circ$ itself

$$\mathcal{L}^E_e e' = e \circ e' \tag{5}$$

for all $e \in \Gamma(E)$ and $e' \in \Gamma(E) = \Gamma(\mathcal{T}_0^{\,1}(E))$. On sections of $\Gamma(E^*)$

$$\langle \mathcal{L}^E_e \alpha, e'\rangle = \mathcal{L}^E_e \langle \alpha, e'\rangle - \langle \alpha, \mathcal{L}^E_e e'\rangle \tag{6}$$

for all $f \in C^\infty(M)$, $e, e' \in \Gamma(E)$ and $\alpha \in \Gamma(E^*) = \Gamma(\mathcal{T}_1^{\,0}(E))$.

Note that one has to use the Leibniz rule in order to guarantee the proper tensorial behavior on the right-hand side of the defining equations. For example, the right-hand side of (6) has to be $C^\infty(M)$-linear in e', which is guaranteed by Leibniz rule. On the other hand, Leibniz identity shows that $e \mapsto \mathcal{L}^E_e$ defines a bracket homomorphism

$$\mathcal{L}^E_{e \circ e'} = \mathcal{L}^E_e \mathcal{L}^E_{e'} - \mathcal{L}^E_{e'} \mathcal{L}^E_e \tag{7}$$

for all $e, e' \in \Gamma(E)$.

In general, one has no relation between $(fe) \circ e'$ and $e \circ e'$. As a consequence, $e \circ e'$ can depend on the values of the section e at every point of the manifold M. If this happens, we cannot restrict the bracket to local sections, which is necessary in order to write it in some local frame components. Hence, in the following we will restrict ourselves only to the so called local Leibniz algebroids, in particular the bracket $\circ$ will be a bidifferential operator of degree one.

We say that the Leibniz algebroid $(E, \rho, \circ)$ is a *local* one,[2] if there exists $L \in \Gamma(\mathcal{T}_2^{\,2}(E))$, such that

$$(fe) \circ e' = f(e \circ e') - (\rho(e').f)e + L(\mathrm{d}f, e, e') \tag{8}$$

where L is viewed as $C^\infty(M)$-trilinear map $L : \Gamma(E^*) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$.

Obviously, L in its first argument is defined uniquely only on the subbundle[3] $\mathrm{Ann}(\ker \rho) \subset E^*$, the annulator of the kernel of the anchor, which is locally generated by sections of the form $\mathrm{d}f$.[4] Nevertheless, using the partition of unity we can define a scalar product on E^* and extend L trivially on the orthogonal complement to $\mathrm{Ann}(\ker \rho)$. Also, the $C^\infty(M)$-trilinearity of L is essential for the definition to be a consistent one.

As a direct consequence of the definition of a local Leibniz algebroid, we see that

$$\rho(L(\mathrm{d}f, e, e')) = 0 \tag{9}$$

i.e., $L(\mathrm{d}f, e, e')$ takes values in the subbundle $\ker \rho$. Moreover, we can always choose an L satisfying $\rho(L(\beta, e, e')) = 0$ for all $\beta \in \Gamma(E^*)$, $e, e' \in \Gamma(E)$, by extending it – as mentioned above – trivially to $(\mathrm{Ann}(\ker \rho))^\perp$.

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1 Not to be confused with ordinary de Rham differential on M.

2 cf. Definition 4.7 and Theorem 4.8 in [9].

3 Assuming that $\ker \rho$ is a subbundle of E.

4 To be more precise, we would have to distinguish between the L defined on $\mathrm{Ann}(\ker \rho) \times \Gamma(E) \times \Gamma(E)$ and its extension to $\Gamma(E^*) \times \Gamma(E) \times \Gamma(E)$, in which case we would consider the L defined on $\mathrm{Ann}(\ker \rho) \times \Gamma(E) \times \Gamma(E)$ being a part of the definition of a local Leibniz algebroid. The choice of an extension would then be an extra datum.