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On computing joint invariants of vector fields

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ABSTRACT

A constructive version of the Frobenius integrability theorem – that can be programmed effectively – is given. This is used in computing invariants of groups of low ranks and recover examples from a recent paper of Boyko et al. (2009).

1. Introduction

The effective computation of local invariants of Lie algebras of vector fields is one of the main technical tools in applications of Lie's symmetry method to several problems in differential equations — notably their classification and explicit solutions of natural equations of mathematical physics, as shown, e.g., in several papers of Ibragimov [1,2], and Olver [3].

The main aim of this paper is to give a constructive procedure that reduces the determination of joint local invariants of any finite dimensional Lie algebra of vector fields – indeed any finite number of vector fields – to that of a commuting family of vector fields. It is thus a constructive version of the Frobenius integrability theorem – [3, p. 422], [4, p. 472], [5, p. 92–94] – which can also be programmed effectively. This is actually valid for any field of scalars. A paper close to this paper is [6].

We illustrate the main results by computing joint invariants for groups of low rank as well as examples from Boyko et al. [7], where the authors have used the method of moving frames, [8], to obtain invariants.

It is stated in [7] that solving the first order system of differential equations is not practicable. However, it is practicable for at least two reasons. The local joint invariants in any representation of a Lie algebra as an algebra of vector fields are the same as those of a commuting family of operators. Moreover, one needs to take only operators that are generators for the full algebra. For example, if the Lie algebra is semisimple with Dynkin diagram having n nodes, then one needs just $2n$ basic operators to determine invariants.

Another reason is that software nowadays can handle symbolic computations very well.

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The main results of the paper are as follows:

Theorem 1. *Let $\mathcal{L}$ be a finite dimensional Lie algebra of vector fields defined on some open subset U of $\mathbb{R}^n$. Let $X_1, \ldots, X_d$ be a basis of $\mathcal{L}$. Then the following hold:*

(1) *The algebra of operators whose coefficient matrix is the matrix of functions obtained from the coefficients of $X_1, \ldots, X_d$ by reducing it to reduced row echelon form is abelian.*
(2) *The local joint invariants of $\mathcal{L}$ are the same as those of the above abelian algebra.*

Theorem 2. *Let $X_1, X_2, \ldots, X_d$ be vector fields defined on some open subset of $\mathbb{R}^n$. Then the joint invariants of $X_1, X_2, \ldots, X_d$ are given by the following algorithm:*

(1) *[Step 1] Find the row reduced echelon form of $X_1, X_2, \ldots, X_d$, and let $Y_1, \ldots, Y_r$ be the corresponding vector fields. If this is a commuting family, then stop. Otherwise go to:*
(2) *[Step 2] If some $[Y_i, Y_j] \neq 0$, then set $Y_{r+1} := [Y_i, Y_j]$. Go to Step 1 and substitute $Y_1, \ldots, Y_r, Y_{r+1}$ in place of $X_1, X_2, \ldots, X_d$.*

This process terminates in at most n iterations. If $V_1, \ldots, V_m$ are the commuting vector fields at the end of the above iterative process, the joint invariants of $X_1, X_2, \ldots, X_d$ coincide with the joint invariants of $V_1, \ldots, V_m$.

2. Some examples and proof of Theorems 1 and 2

Before proving Theorem 1, we give some examples in detail, because these examples contain all the key ideas of a formal proof and of computation of local joint invariants of vector fields.

2.1. Example: The rotations in $\mathbb{R}^3$

The group SO(3) has one basic invariant in its standard representation, namely $x^2+y^2+z^2$, which is clear from geometry. Let us recover this by Lie algebra calculations in a manner that is applicable to all Lie groups.

The fundamental vector fields given by rotations in the coordinate planes are

$$I = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad J = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \quad \text{and} \quad K = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

The coefficients matrix is

$$\begin{pmatrix} y & -x & 0 \\ 0 & z & -y \\ z & 0 & -x \end{pmatrix}. \tag{1}$$

This is a singular matrix, so its rank is at most two. On the open subset U where $yz \neq 0$, the rank is two. The rank is two everywhere except at the origin but we are only interested in the rank on some open set.

The differentiable functions on U simultaneously annihilated by I, J, K are clearly the same as those of the operators whose coefficient matrix is obtained from (1) reducing it to its row echelon form. Since I, J generate the infinitesimal rotations, we may delete the last row in (1). The reduced row echelon form of (1) is

$$\begin{pmatrix} 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \end{pmatrix}.$$

The operators whose matrix of coefficients is this matrix are

$$X := \frac{\partial}{\partial x} - \frac{x}{z}\frac{\partial}{\partial z} \quad \text{and} \quad Y := \frac{\partial}{\partial y} - \frac{y}{z}\frac{\partial}{\partial z}.$$

Note that $[X, Y] = 0$. Now, because the fields are commuting, we can compute the basic invariants of any one of them, say X; then Y will operate on the invariants of X.

The invariants for X are given by the standard method of Cauchy characteristics as follows [5, p. 67]: we want to solve

$$\frac{dx}{1} = \frac{dy}{0} = \frac{-zdz}{x}.$$

The basic invariants of X are $x^2 + z^2 =: \xi$, $y =: \eta$. As Y commutes with X, it operates on invariants of X. Now $Y(\xi) = -2\eta$, $Y(\eta) = 1$. Thus on the invariants of X the field induced by Y is

$$-2\eta\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$