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Centralizers of spin subalgebras

Gerardo Arizmendi, Rafael Herrera *

Centro de Investigación en Matemáticas, A. P. 402, Guanajuato, Gto., C.P. 36000, Mexico

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ABSTRACT

We determine the centralizers of certain isomorphic copies of spin subalgebras $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m)$, where d_r is the dimension of a real irreducible representation of Cl_r^0, the even Clifford algebra determined by the positive definite inner product on $\mathbb{R}^r$, where $r, m \in \mathbb{N}$.

1. Introduction

In this paper, we determine the centralizer subalgebras of (the isomorphic images under certain monomorphisms of) subalgebras $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m)$, where d_r is the dimension of the irreducible representations of Cl_r^0, the even Clifford algebra determined by $\mathbb{R}^r$ endowed with the standard positive definite inner product, and $r, m \in \mathbb{N}$. The need to determine such centralizers has arisen in various geometrical settings such as the following:

- The holonomy algebra of Riemannian manifolds endowed with a parallel even Clifford structure [1].
- The automorphism group of manifolds with (almost) even Clifford (Hermitian) structures [2]. The centralizers determined in this paper help generalize the results on automorphisms groups of Riemannian manifolds [3,4], almost Hermitian manifolds [5], and almost quaternion-Hermitian manifolds [6].
- The structure group of Riemannian manifolds admitting twisted spin structures carrying pure spinors [7]. If M is a smooth oriented Riemannian manifold, F is an auxiliary Riemannian vector bundle of rank r, $S(TM)$ and $S(F)$ are the locally defined spinor vector bundles of M and F respectively, $(f_1, \ldots, f_r)$ is a local orthonormal frame of F, and $m \in \mathbb{N}$ is such that the bundle $S(TM) \otimes S(F)^{\otimes m}$ is globally defined, a *pure spinor field* $\phi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$ is a spinor such that its local 2-forms $\eta_{kl}^{\phi}(X, Y) = \langle X \wedge Y \cdot \kappa_{r*}^{m}(f_k f_l) \cdot \phi, \phi \rangle$ induce at each point $x \in M$ a representation of Cl_r^0 on $T_x M$ without trivial summands. The centralizers determined here are the complements of copies of $\mathfrak{spin}(r)$ in the annihilator algebra of such a spinor. Should the spinor be parallel, such annihilator algebra will contain the holonomy algebra of the manifold and thus be related to the special holonomies of the Berger–Simons holonomy list [8,9].

The paper is organized as follows. In Section 2 we recall some background material and prove three results which will be required later in the main theorems. More precisely, in Section 2.1, we recall standard material about Clifford algebras, Spin groups, Spin algebras, and their representations. In Section 2.2 we find explicit descriptions of the real $\mathfrak{spin}(r)$ representations $\tilde{\Delta}_r$, decompositions into irreducible summands of $\tilde{\Delta}_r \otimes \tilde{\Delta}_r$, and calculate various basic centralizers. In Section 3,

* Corresponding author.
E-mail addresses: gerardo@cimat.mx (G. Arizmendi), rherrera@cimat.mx (R. Herrera).

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we prove the main results of the paper, Theorems 3.1 and 3.2. Namely, in Section 3.1, we find the centralizers of $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m)$ for $r \not\equiv 0 \pmod 4$ (cf. Theorem 3.1) and, in Section 3.2, we find the centralizers of $\mathfrak{spin}(r)$ in $\mathfrak{so}(d_r m_1 + d_r m_2)$ for $r \equiv 0 \pmod 4$ (cf. Theorem 3.2). The proofs involve Riemannian homogeneous spaces, representation theory and Clifford algebras. The separation into two cases is due to the existence of exactly one and two irreducible representations of Cl_r^0 for $r \not\equiv 0 \pmod 4$ and $r \equiv 0 \pmod 4$ respectively.

2. Preliminaries

2.1. Clifford algebra, spin groups and representations

In this section we recall material that can also be consulted in [10,11]. Let Cl_n denote the Clifford algebra generated by all the products of the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad \text{for } 1 \le j, k \le n.$$

We will often write

$$e_{1\ldots s} := e_1 e_2 \cdots e_s.$$

Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C},$$

the complexification of Cl_n. It is well known that

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k \\ \mathrm{End}(\mathbb{C}^{2^k}) \otimes \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k+1, \end{cases}$$

where

$$\mathbb{C}^{2^k} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

the tensor product of $k = [\frac{n}{2}]$ copies of $\mathbb{C}^2$. Let us denote

$$\Delta_n = \mathbb{C}^{2^k},$$

and consider the map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\mathbb{C}^{2^k})$$

which is an isomorphism for n even and the projection onto the first summand for n odd. In order to make κ_n explicit consider the following matrices with complex entries

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Now, consider the generators of the Clifford algebra $e_1, \ldots, e_n$ so that κ_n can be described as follows

$$\begin{aligned}
e_1 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes Id \otimes g_1 \\
e_2 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes Id \otimes g_2 \\
e_3 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes g_1 \otimes T \\
e_4 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes g_2 \otimes T \\
&\vdots \qquad \ldots \\
e_{2k-1} &\mapsto g_1 \otimes T \otimes \cdots \otimes T \otimes T \otimes T \\
e_{2k} &\mapsto g_2 \otimes T \otimes \cdots \otimes T \otimes T \otimes T,
\end{aligned}$$

and the last generator

$$e_{2k+1} \mapsto iT \otimes T \otimes \cdots \otimes T \otimes T \otimes T$$

if $n = 2k+1$.

Let

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i), \qquad u_{-1} = \frac{1}{\sqrt{2}}(1, i)$$

which forms an orthonormal basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Note that

$$g_1(u_{\pm 1}) = iu_{\mp 1}, \qquad g_2(u_{\pm 1}) = \pm u_{\mp 1}, \qquad T(u_{\pm 1}) = \mp u_{\pm 1}.$$