Journal of Geometry and Physics 97 (2015) 119–132

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

A cohomological framework for homotopy moment maps

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ARTICLE INFO

Article history:
Received 16 February 2015
Accepted 3 July 2015
Available online 9 July 2015

Keywords:
Homotopy moment maps
Lie algebras up to homotopy
N-plectic geometry

ABSTRACT

Given a Lie group acting on a manifold M preserving a closed $n+1$-form ω, the notion of homotopy moment map for this action was introduced in Fregier (0000), in terms of L_∞-algebra morphisms. In this note we describe homotopy moment maps as coboundaries of a certain complex. This description simplifies greatly computations, and we use it to study various properties of homotopy moment maps: their relation to equivariant cohomology, their obstruction theory, how they induce new ones on mapping spaces, and their equivalences. The results we obtain extend some of the results of Fregier (0000).

Introduction

Recall that a symplectic form is a closed, non-degenerate 2-form. It is natural to consider symmetries of a given symplectic manifold, that is, a Lie group acting on a manifold, preserving the symplectic form. Among such actions, a nice subclass is given by actions that admit a moment map; in that case the infinitesimal generators of the action are Hamiltonian vector fields. Actions admitting a moment map enjoy remarkable geometric, algebraic and topological properties, that have been studied extensively in the literature (e.g. symplectic reduction, the relation to equivariant cohomology and localization, convexity theorems, etc.).

In this note we consider closed $n+1$-forms for some $n \geq 1$. When they are non-degenerate, they are called multisymplectic form, and are higher analogues of symplectic forms which appear naturally in classical field theory.

Recently Rogers [1] (see also [2]) showed that the algebraic structure underlying a manifold with a closed $n+1$-form ω is one of an L_∞-algebra. This allowed [3] for a natural extension of the notion of moment map to closed forms of arbitrary degree, called *homotopy moment map*. The latter is phrased in terms of L_∞-algebra morphisms.

The first contribution of this note is to construct, out of the action of a Lie group G on a manifold M, a chain complex $\mathcal{C}$ with the following property:

- any invariant closed form ω gives rise to a cocycle $\tilde{\omega}$ in $\mathcal{C}$
- homotopy moment maps are given exactly by the primitives of $\tilde{\omega}$.

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http://dx.doi.org/10.1016/j.geomphys.2015.07.010
0393-0440/

The chain complex $\mathcal{C}$ is simply the product of the Chevalley–Eilenberg complex of the Lie algebra of G, with the de Rham complex of M. The action is encoded by the cocycle $\tilde{\omega}$. Notice that by the above the set of homotopy moment maps (for a fixed ω) has the structure of an affine space, which is unexpected since L_∞-algebra morphisms are generally very non-linear objects.

This characterization of homotopy moment maps is very useful: L_∞-algebra morphisms are usually quite intricate and cumbersome to work with in an explicit way, while working with coboundaries in a complex is much simpler. In this note we use the above characterization to:

- show that certain extensions of ω in the Cartan model give rise to homotopy moment maps (see Section 4),
- give cohomological obstructions to the existence of homotopy moment maps (see Section 5),
- show that a homotopy moment map for a G-action on (M, ω) induces one on $Maps(\Sigma, M)$, the space of maps from any closed and oriented manifold Σ into M, endowed with the closed form obtained from ω by transgression (see Section 6),
- obtain a natural notion of equivalence of homotopy moment maps, both under the requirement that ω be kept fixed and allow ω to vary (see Section 7). We show that it is compatible with the geometric notion of equivalence induced by isotopies of the manifold M, and with the notion of equivalence of L_∞-morphisms (see Appendix A).

In Sections 4 and 5 we obtain results similar to those of [3], but with much less computational effort. The results obtained in Section 7 are a significant extension of results obtained in [3], where only closed 3-forms and loop spaces were considered. The equivalences introduced in 7 and their properties extend and justify the work carried out for closed 3-forms in [3, Section 7.4].

One more application of the characterization of moment maps as coboundaries in $\mathcal{C}$ is the following. Given two manifolds endowed with closed forms, their cartesian product $(M_1 \times M_2, \omega_1 \wedge \omega_2)$ is again an object of the same kind. This construction restricts to the multisymplectic category, but not to the symplectic one. The above characterization of moment maps is used in [4] to construct homotopy moment maps for cartesian products.

Remark. Recall that if $\mathfrak{X}$ is a Lie algebra, a *$\mathfrak{X}$-differential algebra* [5, Section 3] is a graded commutative algebra $\Omega = \oplus_{i\geq 0} \Omega^i$ with graded derivations ι_v, $\mathcal{L}_v$ of degrees -1, 0 (depending linearly on $v \in \mathfrak{X}$) and a derivation d of degree 1 such that the Cartan relations hold:

$$[d, d] = 0 \qquad [\mathcal{L}_v, d] = 0, \qquad [\iota_v, d] = \mathcal{L}_v$$
$$[\iota_v, \iota_w] = 0, \qquad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]_{\mathfrak{X}}}, \qquad [\mathcal{L}_v, \iota_w] = \iota_{[v,w]_{\mathfrak{X}}}.$$

This note is written in terms of geometric objects, but most of it applies also to the algebraic setting obtained replacing the setting we assume in Section 2 with:

$\mathfrak{X}$ a Lie algebra, Ω a $\mathfrak{X}$-differential algebra, $\omega \in \Omega^{n+1}$ with $d\omega = 0$.
$\mathfrak{g}$ a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{X}$ a Lie algebra morphism, so that $\mathcal{L}_{\rho(x)}\omega = 0$
for all $x \in \mathfrak{g}$.

1. Closed forms

We recall briefly how some notions from symplectic geometry apply to closed differential forms of arbitrary degree.

Definition 1.1. Let (M, ω) be a **pre-n-plectic** manifold, i.e., M is a manifold and ω a closed $n+1$-form. An $(n-1)$-form α is **Hamiltonian** iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say v_α is a **Hamiltonian vector field** for α. The set of Hamiltonian $(n-1)$-forms is denoted as $\Omega^{n-1}_{\mathrm{Ham}}(M)$.

In analogy to symplectic geometry, one can endow the set of Hamiltonian $(n-1)$-forms with a skew-symmetric bracket, which however is not a Lie bracket. If one passes from $\Omega^{n-1}_{\mathrm{Ham}}(M)$ to a larger space, one obtains an L_∞-algebra [6], which was constructed essentially in [1, Thm. 5.2], and generalized slightly in [2, Thm. 6.7].

Definition 1.2. Given a pre-n-plectic manifold (M, ω), the **observables** form an L_∞ algebra, denoted $L_\infty(M, \omega) := (L, \{l_k\})$. The underlying graded vector space is given by

$$L_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M) & i = 0, \\ \Omega^{n-1+i}(M) & -n+1 \leq i < 0. \end{cases}$$

The maps $\left\{l_k\colon L^{\otimes k} \to L | 1 \leq k < \infty\right\}$ are defined as

$$l_1(\alpha) = d\alpha,$$