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Analytic connections on Riemann surfaces and orbifolds

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ABSTRACT

We give a differentially closed description of the uniformizing representation to the analytical apparatus on Riemann surfaces and orbifolds of finite analytic type. Apart from well-known automorphic functions and Abelian differentials it involves construction of the connection objects. Like functions and differentials, the connection, being also the fundamental object, is described by algorithmically derivable ODEs. Automorphic properties of all of the objects are associated to different discrete groups, among which are excessive ones. We show, in an example of the hyperelliptic curves, how can the connection be explicitly constructed. We study also a relation between classical/traditional 'linearly differential' viewpoint (principal Fuchsian equation) and uniformizing τ-representation of the theory. The latter is shown to be supplemented with the second (to the principal) Fuchsian equation.

1. Introduction

Riemann surfaces are of fundamental importance to the mathematical physics because most effective part of the modern differential/integral calculus is related, one way or the other, with a complex analysis on a certain Riemann surface of (or not) a certain analytic function. Surfaces of a finite genus are distinctive in that the calculus is the best elaborated one with lot of applications. Strangely enough, the standard differential apparatus on such kind objects of higher genera ($g > 1$) cannot be considered as completely closed; this remark requires some explanation.

Let $\mathscr{R}$ be a finite genus Riemann surface determined by irreducible algebraic equation

$$F(x, y) = 0. \tag{1}$$

Uniformizing representation of $\mathscr{R}$ is given by a pair of single-valued analytic functions $x = \varphi(\tau)$, $y = \psi(\tau)$, wherein the global uniformizer τ belongs to the upper half-plane $\mathbb{H}^+$, that is $\Im(\tau) > 0$. Functions φ and ψ are the automorphic ones with respect to an infinite discrete Fuchsian group $\mathfrak{G}_{\mathscr{R}}$ [1]:

$$\varphi\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \varphi(\tau), \qquad \psi\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \psi(\tau), \qquad \forall \tau \in \mathbb{H}^+, \tag{2}$$

where $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathfrak{G}_{\mathscr{R}} \subset \mathrm{PSL}_2(\mathbb{R})$. Complex analysis on $\mathscr{R}$ includes Abelian differentials $R(x, y)dx$, their integrals $\int R(x, y)dx$, and the $\mathscr{R}$ itself is completely determined by periods of Abelian integrals that are holomorphic (everywhere finite) [2]. We know also that if some function $\psi(\tau)$ is a τ-representation for any of the differentials above then its automorphic property is characterized by a weight-2 automorphic form [1,3,2]:

$$\psi\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = (\gamma\tau+\delta)^2 \cdot \psi(\tau). \tag{3}$$

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In the uniformization theory automorphic functions are described by the 2nd order linear ordinary differential equations (ODEs) of a Fuchsian class [1]

$$\Psi_{xx} = \frac{1}{2}\mathcal{Q}(x, y)\Psi, \tag{4}$$

where $\mathcal{Q}$ is a rational function of its arguments. In general, this is a Fuchsian equation with algebraic coefficients. The uniformizing parameter τ is then defined as a ratio

$$\tau = \frac{\Psi_2(x)}{\Psi_1(x)} \tag{5}$$

of the linearly independent solutions to Eq. (4) and inversion of this ratio determines one of the uniformizing functions: $x = \wp(\tau)$. It is well known that the theory is equivalent to the 3rd order ODE $\{\tau, x\} = -\mathcal{Q}(x, y)$ containing no the auxiliary Ψ-function, where $\{\tau, x\}$ is the standard notation for the Schwarz derivative [1]

$$\{\tau, x\} := \frac{\tau_{xxx}}{\tau_x} - \frac{3}{2}\frac{{\tau_{xx}}^2}{{\tau_x}^2}. \tag{6}$$

Since the theory is described by the third order ODEs, its complete data set is not exhausted by functions (2) and their first order differentials (3): the second order differentiation is missing. Alternatively, the $\mathscr{R}$ may be thought of as a 1-dimensional complex manifold [3] and all the objects above can be treated from the differential geometric viewpoint. Then functions (2)–(3) represent scalars and 1-forms and calculus should involve a covariant differentiation of these and other tensor fields. By this means, in order to close the complex analytic theory, we have to introduce (at least) a canonical bundle over our $\mathscr{R}$ and corresponding connection object Γ. Partially, some ingredients of such a view on the theory have already been appeared in the literature. Dubrovin [4] gave a geometric treatment to the famous Chazy equation $\pi\dddot{\eta} = 12\mathrm{i}(2\eta\ddot{\eta} - 3\dot{\eta}^2)$ when the group $\mathfrak{G}_{\mathscr{R}}$ is the genus zero full modular group $\mathrm{PSL}_2(\mathbb{Z}){=:}\boldsymbol{\Gamma}(1)$ and Hawley & Schiffer [5] introduced the connection Γ in the context of conformal mappings of planar domains and multi-connected representations of $\mathscr{R}$. The well-known modular forms [6] are the particular cases of automorphic forms when group is a subgroup of $\boldsymbol{\Gamma}(1)$. They possess interesting differential properties and some of them – ODEs for some low level groups $\Gamma_0(N)$ – are constructed in [7]. It may be remarked here that even the theory of the $\boldsymbol{\Gamma}(1)$-connection function, i.e., the Chazy–Weierstrass function $\eta(\tau)$, is not restricted by the Chazy equation mentioned above. Recent work [8] provides an alternative theory (and nontrivial application) in the language of linear Fuchsian equations (4).

The known examples [4,7,6] are concerned only with the zero genus cases and general automorphic properties of bundles and connections on them, to our knowledge, are not considered in the literature. This is the subject matter of the present work. We give an analytically closed geometric description for the differential calculus on Riemann surfaces of finite analytic type (genus and number of punctures are finite) through the uniformizing τ-representation for the connection objects $\Gamma(\tau)$ and characterize their differential properties. More precisely, not only do functions and differentials satisfy some autonomic 3rd order ODEs (the known fact [1,9,6]), but connections also satisfy equations of such a kind. What is more, a remarkable property of (analytic) connections on $\mathscr{R}$'s of arbitrary genera is the fact that all of them come from a trivial connection on certain orbifolds of the *zero* genus and satisfy autonomic ODEs. These ODEs are algorithmically derivable.

2. Invariant quantities on $\mathscr{R}$

2.1. Invariant counterparts of Fuchsian equations

Let Fuchsian equation (4) determine, through its monodromical group $\mathfrak{G}_x$, an exact representation of fundamental group π_1 of a certain orbifold or the $\mathscr{R}$ itself. However, from differential geometric viewpoint this equation is not well defined; it has no invariant (autonomic) form. Indeed, it contains explicitly the quantity x which, in a generic case $F_y(x, y) \neq 0$, is the standard usage for a local coordinate on $\mathscr{R}$. In turn the quantity $\tau(x)$ coming from the definition (5) obviously does not produce the geometric object. However we may take 1-dimensionality of $\mathscr{R}$ into account and swap around the standard coordinate x and 'object' τ; thus x may be thought of as the scalar 'field' quantity $x = x(\mathscr{P})$ [10] being represented by function $x = \wp(\tau)$ on the universal cover $\mathbb{H}^+$. Then the automorphic property (2) becomes nothing but the $\mathbb{H}^+/\mathfrak{G}$-factor topology reformulation to the property of x to be a scalar:

$$\tilde{x}(\mathscr{P}) = x(\mathscr{P}), \qquad \wp\Big(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big) = \wp(\tau); \tag{7}$$

here $\tilde{x}(\mathscr{P})$ is a value of the quantity x at point $\mathscr{P} \in \mathscr{R}$ under the coordinate choice[1]

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d} \tag{8}$$

[1] Greek symbols $\alpha, \beta, \ldots$ will be used for discrete group transformations and Latin $a, b, \ldots$ for coordinate changes.