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Gauge symmetries in 2D field theory

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ABSTRACT

A simple algorithm is proposed for constructing generators of gauge symmetry as well as reducibility relations for arbitrary systems of partial differential equations in two dimensions.

1. Introduction

In this paper we suggest a simple algorithm for finding all gauge symmetries, given a system of local field equations in two dimensions. The method works equally well for Lagrangian and non-Lagrangian equations and it is local in space–time. In contemporary field theory, the field equations are often constructed with a pre-specified gauge symmetry. In that case, one has to be sure that the theory does not have any other gauge symmetry. So, a systematic method of identifying a *complete* gauge symmetry of given field equations can be useful even for the models having some known gauge invariance by construction.

The Dirac–Bergmann algorithm allows one to find all gauge symmetries for Lagrangian dynamics by casting the equations into the normal form of the constrained Hamiltonian formalism [1–3]. This algorithm can be extended to the general systems, not necessarily Lagrangian, by bringing the dynamics to the involutive normal form [4]. The Dirac–Bergmann algorithm was originally formulated for mechanical systems. In this form, it has been further developed by most of the followers, see for review [2–4]. Its extension to field theory is straightforward if the locality in space is not an issue.[1] Besides locality, the other subtleties are known concerning application of the classical Dirac–Bergmann algorithm to field theories [5].

The explicit knowledge of space–time local generators of a complete gauge symmetry is a necessary pre-requisite for solving most of crucial problems in field theory, like identifying global symmetries and conservation laws, constructing consistent interactions and quantization [6]. The recent developments in the BRST formalism [7–10] allow one to solve the same range of problems for not necessarily Lagrangian field theories. The list of examples of non-Lagrangian models of current interest includes chiral bosons in various dimensions, Seiberg–Witten and Donaldson–Uhlenbeck–Yau equations, various conformal field theories with extended supersymmetry, and Vasiliev's equations of interacting higher-spin massless fields.

While the importance of explicit identification of gauge symmetries is widely recognized in physics, on the mathematical side the gauge invariance of PDEs is often considered as an "unpleasant complication", which should be overcome

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[1] For instance, the Dirac bracket, being an important part of the Dirac formalism, may be non-local in space [2,3].

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immediately by imposing appropriate gauge fixing conditions making the system fully determined (see e.g. [11,12]). Perhaps, the only exception to this practice is the mathematical theory of optimal control [13], where the gauge symmetry reincarnates as controllability. An expanded discussion of the relationship between both the concepts can be found in [4]. In that paper, we also described a normal form that the general system of ODEs can be brought into, and proved some basic theorems on the structure of gauge symmetry transformations.

The present paper extends the results of [4] to the general, not necessarily Lagrangian, 2D field theory, providing a systematic method for finding a complete gauge symmetry. The extension is not straightforward due to the appearance of new integrability conditions steaming from the commutativity of partial derivatives, that has no analogue in mechanics. The main difference, however, is the change of the ground ring underlying the analysis of gauge symmetries. The situation can be described schematically by the following table:

	ODEs : $D = 1$	PDEs, $D = 2$	PDEs, $D > 2$
The ground ring	meromorphic functions (m.f.)	ordinary diff. operators with coefficients in m.f.	partial diff. operators with coefficients in m.f.
Algebraic properties	commutative, differential field	non-commutative, principal ideal domain	non-commutative, Noetherian

As is seen, the 2D field theories are intermediate in algebraic properties between ODEs and higher dimensional PDEs. This allows us to consider the case of two dimensions as special.[2]

The paper is organized to follow the structure of the algorithm we propose for finding gauge symmetries. The algorithm includes three steps. Given a system of 2D PDEs, we transform it to the Cartan normal form, revealing all the hidden integrability conditions, if any. As a result we get an involutive system of the first-order PDEs. This preparatory step is quite standard and it is explained in Section 2. In the same section, we also recall an algebraic background needed for a rigorous definition of the notion of a gauge symmetry and illustrate this notion by two simple yet general examples. In Section 3.1, we show that any 2D field theory can be embedded into a constrained Hamiltonian system, which we call the Pontryagin system, in such a way that the gauge symmetries of the original equations extend to those of the Pontryagin action. Applying the second Noether theorem to the Pontryagin system reduces the problem of finding gauge symmetries to that of finding the Noether identities, as it is explained in Section 3.2. Due to the special structure of the Hamiltonian equations, the latter problem amounts to constructing differential identities among the primary Hamiltonian constraints and it is the point where the theory of finitely generated modules over rings of differential operators comes to forefront. In Section 3.3, we construct a minimal free resolution for the differential module associated with the Hamiltonian constraints, from which both a generating set for the gauge symmetry transformations and the corresponding reducibility relations can be read off. Among other things, this construction provides a direct proof of the fact that in 2D field theory any gauge symmetry admits no more than one stage of reducibility.

In Section 4, we consider a particular example of nonlinear relativistic field equation. As well as being an illustration to our method, it demonstrates an interesting phenomenon of bifurcation of the structure of gauge symmetry when one varies numerical parameters entering the model. In particular, it shows that a smooth deformation of free field equations by inclusion of interaction is not always followed by a smooth deformation of the corresponding gauge generators, even though the overall number of independent gauge symmetries is preserved.

In the concluding Section 5, we summarize our results and formulate two plausible conjectures about the count of physical degrees of freedom in 2D field theory. Appendix contains a useful theorem on the matrices over the ring of ordinary differential operators.

2. Cartan normal form and gauge symmetries

By a two-dimensional field theory we understand an arbitrary system of PDEs with two independent variables. We fix neither the order of equations nor their number, which may be completely arbitrary and in no way correlate with the number of dependent variables (fields). In this section, we discuss a normal form each two-dimensional system of field equations can be brought into at the cost of introducing auxiliary fields. This normal form will be a starting point for the study of gauge symmetries in the next section.

2.1. Pfaffian systems

Let $\Lambda(M) = \bigoplus \Lambda^k(M)$ denote the exterior algebra of differential forms on an n-dimensional smooth manifold M and let $\mathcal{I} \subset \Lambda(M)$ be an ideal of $\Lambda(M)$. A submanifold $\Sigma \subset M$ is called an integral manifold of $\mathcal{I}$ if $\alpha|_\Sigma = 0$ for all $\alpha \in \mathcal{I}$. In the case where the ideal $\mathcal{I}$ is generated by a set of 1-forms Θ^J and 0-forms Φ_A the looking-for integral manifolds is known as

[2] Let us also mention a plenty of nonlinear integrable models known in $D = 2$, though this fact is not directly related to the present work.