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A holographic principle for the existence of imaginary Killing spinors

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ABSTRACT

Suppose that $\Sigma = \partial\Omega$ is the n-dimensional boundary, with positive (inward) mean curvature H, of a connected compact $(n+1)$-dimensional Riemannian spin manifold (Ω^{n+1}, g) whose scalar curvature $R \geq -n(n+1)k^2$, for some $k > 0$. If Σ admits an isometric and isospin immersion F into the hyperbolic space $\mathbb{H}^{n+1}_{-k^2}$, we define a quasi-local mass and prove its positivity as well as the associated rigidity statement. The proof is based on a holographic principle for the existence of an imaginary Killing spinor. For $n = 2$, we also show that its limit, for coordinate spheres in an Asymptotically Hyperbolic (AH) manifold, is the mass of the (AH) manifold.

1. Introduction

The Positive Mass Theorem (PMT) states that for a complete asymptotically flat manifold which, near each end, behaves like the Euclidean space at infinity and whose scalar curvature is nonnegative, its ADM mass of each end is non-negative. Moreover, if the ADM mass of one end is zero, then the manifold is the Euclidean space. The PMT was proved by Schoen and Yau [1,2] using minimal surface techniques. Later on, Witten [3] gave an elegant and simple proof of the PMT for spin manifolds. Since then, spinors have been successfully used to prove Positive Mass type theorems (see for example [4–11]).

In this spirit, Wang and Yau [12] introduced a quasi-local mass for 3-dimensional manifolds with boundary whose scalar curvature is bounded from below by a negative constant. Again, using spinorial methods, they proved that this mass is non-negative. Shi and Tam [13] proved a similar result but with a simpler and more explicit definition of the mass. More precisely:

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Theorem 1. [13] *Let (Ω^3, g) be a compact 3-dimensional orientable Riemannian manifold with smooth boundary Σ. Assume that:*

(1) *The scalar curvature R of (Ω, g) satisfies $R \geq -6k^2$ for some $k > 0$;*
(2) *The boundary Σ is a topological sphere with Gauss curvature $K > -k^2$ and mean curvature $H > 0$ (so that Σ can be isometrically embedded into $\mathbb{H}^3_{-k^2}$, the Hyperbolic space of constant curvature $-k^2$, with mean curvature H_0).*

Then, the energy–momentum vector

$$\mathcal{M}_\alpha := \int_\Sigma (H_0 - H)\mathbf{W}_\alpha \, d\Sigma \in \mathbb{R}^{3,1}$$

is future directed non-spacelike or zero, where $\mathbf{W}_\alpha = (x_1, x_2, x_3, \alpha t)$ with

$$\alpha = \coth R_1 + \frac{1}{\sinh R_1}\Big(\frac{\sinh^2 R_2}{\sinh^2 R_1} - 1\Big)^{\frac{1}{2}} > 1 \tag{1}$$

an explicit constant depending on the intrinsic geometry of Σ and $\mathbf{X} := \mathbf{W}_1 = (x_1, x_2, x_3, t)$ is the position vector in $\mathbb{R}^{3,1}$. Moreover, if there exists a future directed null vector $\zeta \in \mathbb{R}^{3,1}$ such that:

$$\langle \mathcal{M}_\alpha, \zeta \rangle_{\mathbb{R}^{3,1}} = 0,$$

then (Ω^3, g) is a domain in $\mathbb{H}^3_{-k^2}$.

The statement of this result needs some explanation. First, from [14,15], as mentioned, the assumptions on the boundary Σ ensure the existence of an isometric embedding of Σ into the hyperbolic space $\mathbb{H}^3_{-k^2}$ as a convex surface which bounds a domain D in $\mathbb{H}^3_{-k^2}$. Moreover, this embedding is unique up to an isometry of $\mathbb{H}^3_{-k^2}$. Here H_0 denotes the mean curvature of this embedding and R_1 and R_2 are two positive real numbers such that $B_o(R_1) \subset D \subset B_o(R_2)$ in $\mathbb{H}^3_{-k^2}$ where $B_o(r)$ is the geodesic ball of radius $r > 0$ and center $o = (0, 0, 0, 1/k)$. This result has been recently generalized by Kwong [16]. Namely, he proves:

Theorem 2. [16] *For $n \geq 2$, let (Ω^{n+1}, g) be a compact spin $(n+1)$-dimensional manifold with smooth boundary Σ. Assume that:*

(1) *The scalar curvature R of Ω satisfies $R \geq -n(n+1)k^2$ for some $k > 0$,*
(2) *The boundary Σ is topologically an n-sphere with sectional curvature $K > -k^2$, mean curvature $H > 0$ and that Σ can be isometrically embedded uniquely into $\mathbb{H}^{n+1}_{-k^2}$ with mean curvature H_0.*

Then, there is a future time-like vector-valued function $\mathbf{W}_\alpha$ on Σ such that the energy–momentum vector:

$$\mathcal{M}_\alpha := \int_\Sigma (H_0 - H)\mathbf{W}_\alpha \, d\Sigma \in \mathbb{R}^{n+1,1}$$

is future non-spacelike. Here $\mathbf{W}_\alpha = (x_1, x_2, \ldots, x_{n+1}, \alpha t)$ for some $\alpha > 1$ and $\mathbf{X} := \mathbf{W}_1(x_1, x_2, \ldots, x_{n+1}, t) \in \mathbb{H}^{n+1}_{-k^2} \subset \mathbb{R}^{n+1,1}$ is the position vector of the embedding of Σ.

In the general case, the constant α is still explicitly given by (1). It is conjectured, and verified for $n = 2$ in certain cases (see [13]), that Theorems 1 and 2 should hold for $\alpha = 1$. A key ingredient in the proof of these two results is a generalization of the Positive Mass Theorem for (AH) manifolds (see Section 4.3).

In this paper, we make use of another approach, developed in [11], to establish a holographic principle[1] for the existence of imaginary Killing spinors on Dirac bundles (See Section 2.2) in order to generalize the above results in several directions. Namely, we modify the curvature term in the definition of $\mathcal{M}_\alpha$ to precisely define an energy–momentum vector field $\mathbf{E}(\Sigma)$ in terms of $\mathbf{X}$. In particular, our expression depends only on the metric and the embedding of Σ and is thus independent of the particular manifold Ω. It could be considered as a possible new definition of a quasi-local mass since it has the desirable *non negativity* and *rigidity* properties as shown in Theorems 3 and 4. Moreover, these statements hold in a more general setup.

In fact, we have:

Theorem 3. *Let (Ω^{n+1}, g) be a compact, connected $(n+1)$-dimensional Riemannian spin manifold with smooth boundary Σ. Assume that*

(1) *The scalar curvature R of Ω satisfies $R \geq -n(n+1)k^2$ for some $k > 0$;*
(2) *The boundary $\Sigma = \partial\Omega$ has mean curvature $H > 0$ and that there exists an isometric and isospin immersion F of Σ into the hyperbolic space $\mathbb{H}^{n+1}_{-k^2}$ with mean curvature H_0.*

[1] By holographic principle we mean the property which states that the description of a manifold with boundary can be thought of as encoded on the boundary.