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Geodesics and submanifold structures in conformal geometry

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ABSTRACT

A conformal structure on a manifold M^n induces natural second order conformally invariant operators, called Möbius and Laplace structures, acting on specific weight bundles of M, provided that $n \geq 3$. By extending the notions of Möbius and Laplace structures to the case of surfaces and curves, we develop here the theory of extrinsic conformal geometry for submanifolds, find tensorial invariants of a conformal embedding, and use these invariants to characterize various notions of geodesic submanifolds.

1. Introduction

The existence of a unique covariant derivative makes differential calculus, including the concept of totally geodesic submanifolds (or, more generally, the tensorial invariants of a Riemannian embedding) in a Riemannian manifold straightforward.

On a conformal manifold, where there is no such canonical covariant derivative (there is, however, a *Cartan connection* on an enlarged bundle, for dimension at least 3, [1]), a concept of *conformal geodesics* is also given [2,3] starting from dimension 3 onwards.

These *conformal geodesics* are curves that are solutions of a 3rd order ODE that depends on the conformal structure alone.

In this paper, we intend to characterize higher-dimensional submanifolds that fulfill some *geodesic* properties in the conformal setting and describe the geometric properties and invariants of a conformal embedding.

To make the theory fully general, a first inconsistency of conformal geometry has to be overcome: indeed, while in dimensions larger than 3 a conformal manifold admits an associated Cartan connection and is, therefore, *rigid*, on curves, a conformal structure means just a differential structure, and on surfaces, a conformal structure and an orientation are equivalent to a complex structure—both are examples of *flexible structures*.

Here, we call a geometric structure on M *rigid* if on every open set, the dimension of the space of infinitesimal transformations (vector fields) preserving the given structure is bounded by a number that depends only on dim M and the corresponding structure. Otherwise it is *flexible*. Examples of rigid structures are Riemannian metrics, conformal structures if

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dim $M \geq 3$, *CR* structures and, more generally, all structures that admit a canonical Cartan connection; symplectic, complex, contact structures are examples of flexible structures.

Using the concept of a *Möbius structure*, defined by D. Calderbank as a linear second order differential operator of a certain type [4], and also using a *Laplace structure* (a variant of the *conformal Laplacian*) to rigidify a curve, we create a setting conformal-Möbius–Laplace for which the questions of submanifold geometry can be studied without conditions on the dimension (see also [5]).

In particular, on a Möbius surface or a Laplace curve, the concept of a conformal (or rather Möbius, resp. Laplace) geodesic is well-defined, and denotes, as well, the family of curves that are solutions to a 3rd order ODE. Technically, these equations are given, in terms of a conformal covariant derivative (*Weyl structure*) and of its associated *Schouten–Weyl tensor*, and this tensor is defined, in low dimensions, precisely by the corresponding additional structure (Möbius, resp. Laplace) [5]. The invariants of, and induced structures on a conformal embedding are also defined in terms of these Schouten–Weyl tensors, the distinction between them being the following:

- An intrinsic structure is one that can be defined and considered in terms of the submanifold alone, without any reference to the embedding: it is the case of the induced conformal, Möbius and Laplace structures [5].
- An extrinsic kind of structure refers explicitly to (some infinitesimal version – like the normal bundle – of) the embedding of the submanifold in its conformal (or Möbius) ambient space: it is the case of some tensorial invariants of the embedding and of the induced connection on the weightless normal bundle.

Geometrically, a Laplace structure on a curve is a *projective structure* [6,7,5], and the global projective geometry of a closed curve in a conformal (or Möbius) ambient space turns out to be a very interesting, and largely unknown problem, as a forthcoming paper shows [8].

After a preliminary section where we recall some basic facts of conformal geometry (weight bundles, Weyl structures, and curvature decompositions), with a particular focus on low dimensions, we review in Section 3 the theory of *Möbius* and *Laplace* structures [5] on conformal manifolds, structures that are rigid in small dimensions as well.

Section 4 (Extrinsic conformal geometry) is divided into 3 parts: in Section 4.1 we recall the definition and properties of the conformal geodesic equation [2,3]. In Section 4.2 we review the relation between Weyl structures on submanifolds and on the ambient space [3,5], and introduce certain tensorial invariants of a conformal embedding: the (well-known) trace-free fundamental form, the curvature of the normal bundle and the *mixed* and the *relative* Schouten–Weyl tensor. We show in Theorem 4.22 that any given such tensorial objects on a given manifold and on its normal bundle can be realized as the invariants of an embedding in some ambient space. In Section 4.3, we review the induced Möbius and Laplace structures on a submanifold (the *Möbius reduction* of [5]) and relate them to the previously introduced relative Schouten–Weyl tensor.

In Section 5, we show that the invariant tensors (from Section 4.2) of an embedding turn out to be obstructions for various properties that generalize, in the conformal context, the *totally geodesic submanifolds* of Riemannian geometry.

More precisely, a submanifold is called *totally umbilic* if it is totally geodesic for some metric in the conformal class, it is *weakly geodesic* if it is spanned by conformal geodesics in the ambient space, and *strongly geodesic* if its conformal geodesics are also conformal geodesics in the ambient space (for dimensions 1 or 2, the conformal structure of the (sub)manifold needs to be completed (for rigidity) by a Laplace, resp. Möbius structure).

Finally, these different kinds of geodesic properties of a submanifold are shown to satisfy some implications (among which the fact that *strongly geodesic* implies *weakly geodesic* turns out to be non-trivial), and can be characterized by the vanishing of some of the above mentioned tensorial invariants, Theorem 5.4.

2. Preliminaries on conformal geometry

In this section, we review the main notions needed in conformal geometry. Good references are [9,10], however we need to push some of the formulas beyond their usual lower bound for the dimension, like in [4] (in particular for the Schouten–Weyl tensor and the normalized scalar curvature); a reader familiar with the formalism of weight bundles, Weyl structures, etc., may jump directly to Proposition 2.11.

2.1. Weight bundles

Let M be a m-dimensional manifold with density bundle $|\Lambda|M$. This is an oriented line bundle, hence topologically (but non-canonically) trivial, whose positive sections are the *volume elements* of M, allowing the integration of functions on the manifold; it is isomorphic, if M is oriented, with $\Lambda^m M$, the bundle of m-forms on M (the isomorphism depends on the orientation). Denote by L the dual (or inverse) of the mth root of $|\Lambda|M$, thus $|\Lambda|M \simeq L^{-m}$.

A *conformal structure* on M is a *positive-definite* symmetric bilinear form c on TM with values in the line bundle $L^2 := L\otimes L$, or, equivalently, a non-degenerate section $c \in C^\infty(S^2M \otimes L^2)$ (here we denote by S^2M the bundle of symmetric bilinear forms on TM), with the following normalization condition:

$$|\det c| : (\Lambda^m TM)^2 \to (L^2)^m$$

is the identity. (Note that $(\Lambda^m M)^2 \simeq (|\Lambda|M)^2 \simeq L^{2m}$.)