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Torus action on the moduli spaces of torsion plane sheaves of multiplicity four

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ABSTRACT

We describe the torus fixed locus of the moduli space of stable sheaves with Hilbert polynomial $4m+1$ on $\mathbb{P}^2$. We determine the torus representation of the tangent spaces at the fixed points, which leads to the computation of the Betti and Hodge numbers of the moduli space.

1. Introduction

Let $\mathrm{M} = \mathrm{M}_{\mathbb{P}^2}(4, 1)$ denote the moduli space of Gieseker semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ having Hilbert polynomial $P_{\mathcal{F}}(m) = 4m + 1$. According to [1], M is an irreducible smooth projective variety of dimension 17. Our aim is to classify the torus fixed locus of M under the natural torus action induced from the torus action on the base space $\mathbb{P}^2$, which in turn enables us to compute the additive structure of its homology groups with coefficients in $\mathbb{Z}$. For this we will use the theory of Białynicki-Birula [2–4], which we review in Section 2.

More precisely, we will consider the natural action of $T = (\mathbb{C}^*)^2$ defined as follows: first, T acts on $\mathbb{P}^2$ by $(t_1, t_2) \cdot (x_0, x_1, x_2) = (x_0, t_1^{-1}x_1, t_2^{-1}x_2)$; denote by $\mu_t \colon \mathbb{P}^2 \to \mathbb{P}^2$ the action map. Now put $t[\mathcal{F}] = [\mu_{t^{-1}}^* \mathcal{F}]$, where $[\mathcal{F}]$ denotes the stable-equivalence class of the sheaf $\mathcal{F}$. For this action we will prove the following theorem:

Theorem 1.1. *The fixed point locus of* $\mathrm{M}_{\mathbb{P}^2}(4, 1)$ *consists of* 180 *isolated points and* 6 *one-dimensional components isomorphic to* $\mathbb{P}^1$. *Furthermore, the integral homology of* $\mathrm{M}_{\mathbb{P}^2}(4, 1)$ *has no torsion and its Poincaré polynomial is*

$$P_{\mathrm{M}_{\mathbb{P}^2}(4,1)}(x) = 1 + 2x^2 + 6x^4 + 10x^6 + 14x^8 + 15x^{10} + 16x^{12} + 16x^{14} + 16x^{16} + 16x^{18} + 16x^{20} + 16x^{22} + 15x^{24} + 14x^{26} + 10x^{28} + 6x^{30} + 2x^{32} + x^{34}.$$

The geometry of the moduli space M has been studied by many authors [5–8]. In [9], it was conjectured that genus zero Gopakumar–Vafa (or BPS) invariant defined in M-theory is equal to the Euler characteristic of the moduli space $\mathrm{M}_{\mathbb{P}^2}(r, 1)$

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up to sign. When $r \leq 3$, the moduli spaces are well known by the work of Le Potier [1]. When $r = 4$, the conjecture was first checked in [7] where the author uses a stratification of the moduli space with respect to the global section spaces. The Poincaré polynomials of the moduli spaces when $r = 4$ and 5 have been computed in [5] by a wall-crossing technique in the moduli spaces of α-stable pairs, and also in [8] by the classification of the semi-stable sheaves carried out in [6,10]. Recently, a B-model calculation in physics computes the Poincaré polynomial up to $r = 7$ [11, Table 2] in terms of the refined BPS indices. From a mathematical point of view this calculation is a conjecture. A more mathematical treatment for the refined BPS index can be found in [12]. The Poincaré polynomial in Theorem 1.1 agrees with all these previous works.

We will use two approaches to determine M^T. In Section 3, following [13], we will regard a T-fixed sheaf as a T-equivariant sheaf and we will classify all T-equivariant sheaves in terms of T-representations on each affine open subset. In Section 5, we will use the classification of semi-stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4m + 1$ carried out in [6].

The Poincaré polynomial can then be computed by analyzing the local structure of the moduli spaces around the fixed locus, that is, by determining the T-action on the tangent spaces at the fixed points. For this, we will use two approaches as well. In Section 4, we will use the technique developed in [14]. Using the T-representations of sheaves on each affine open subset and the associated Čech complex, we compute the T-representation of the tangent spaces. In Section 6, we will exploit the locally closed stratification of $\mathrm{M}_{\mathbb{P}^2}(4, 1)$ found in [6].

The first approach can, in principle, be applied to any non-singular moduli space $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$ of semi-stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $rm + \chi$, though, of course, for higher multiplicity the calculations will be considerably more involved. The second approach can be extended to semi-stable sheaves supported on plane quintics or sextics, for which the classification has been carried out, cf. [10] and [15].

Theorem 1.1 also allows us to compute the Hodge numbers h^{pq} of $\mathrm{M}_{\mathbb{P}^2}(4, 1)$. According to [16, Theorem 1], for any one-parameter subgroup λ of T, $h^{pq} = 0$ if $|p-q| > \dim(\mathrm{M}^\lambda)$. Choosing a generic λ we see that $h^{pq} = 0$ if $|p-q| > \dim(\mathrm{M}^T) = 1$. We obtain the following:

Proposition 1.2. *The Hodge numbers h^{pq}, $0 \leq p, q \leq 17$, of $\mathrm{M}_{\mathbb{P}^2}(4, 1)$ satisfy the relations $h^{pq} = 0$ if $p \neq q$ and $h^{pp} = b_{2p}$, where b_{2p} is the Betti number obtained in Theorem 1.1.*

2. A review of the Białynicki-Birula theory

As a preliminary, we briefly review the theory of Białynicki-Birula which will be used throughout the paper. Let X be a smooth projective variety with an action of a torus T. As usual, we denote by M the group of characters of T, and by N the group of one-parameter subgroups of T. Consider $\lambda \in N$ and the associated $\mathbb{C}^*$-action on X defined by $(t, x) \mapsto \lambda(t) \cdot x$. Let $X_1^\lambda, \ldots, X_r^\lambda$ denote the irreducible components of the $\mathbb{C}^*$-fixed point locus X^λ. They are smooth subvarieties. We have a *plus decomposition*

$$X = X_1^{\lambda+} \cup \cdots \cup X_r^{\lambda+}$$

of X into *plus cells*

$$X_i^{\lambda+} = \{x \in X \mid \lim_{t\to 0} \lambda(t) \cdot x \in X_i^\lambda\}.$$

Analogously, we have a *minus decomposition* of X into *minus cells*

$$X_i^{\lambda-} = \{x \in X \mid \lim_{t\to \infty} \lambda(t) \cdot x \in X_i^\lambda\}.$$

The plus and minus cells are bundles on X^λ. More precisely, for each i the restricted tangent bundle $\mathrm{T}_{X|X_i^\lambda}$ can be decomposed as a direct sum of sub-bundles,

$$\mathrm{T}_{X|X_i^\lambda} = \mathrm{T}_i^+ \oplus \mathrm{T}_i^0 \oplus \mathrm{T}_i^-,$$

on which $\mathbb{C}^*$ acts with positive, zero, respectively negative weights. Denote $p(i) = \mathrm{rank}(\mathrm{T}_i^+)$, $n(i) = \mathrm{rank}(\mathrm{T}_i^-)$. Then $X_i^{\lambda+}$ (respectively $X_i^{\lambda-}$) is a fiber bundle over X_i^λ whose fiber is an affine space of dimension $p(i)$ (respectively $n(i)$). From the plus and minus decompositions we obtain the Homology Basis Formula [4, Theorem 4.4]:

Theorem 2.1. *For any integer m with $0 \leq m \leq 2\dim(X)$, we have a decomposition*

$$\mathrm{H}_m(X, \mathbb{Z}) \simeq \bigoplus_{1\leq i\leq r} \mathrm{H}_{m-2p(i)}(X_i^\lambda, \mathbb{Z}) \simeq \bigoplus_{1\leq i\leq r} \mathrm{H}_{m-2n(i)}(X_i^\lambda, \mathbb{Z}).$$

Thus, the Poincaré polynomial P_X of X satisfies the relation

$$P_X(x) = \sum_{i=1}^r P_{X_i^\lambda}(x) x^{2p(i)} = \sum_{i=1}^r P_{X_i^\lambda}(x) x^{2n(i)}.$$

According to [4, Lemma 4.1], for generic $\lambda \in N$ we have $X^T = X^\lambda$. In fact, this is true for all λ satisfying the condition: $\langle \lambda, \chi\rangle \neq 0$ for all $\chi \in M$ occurring in the weight decomposition of any tangent space $\mathrm{T}_{X,x}$ at a fixed point $x \in X^T$. Thus,