Journal of Geometry and Physics 83 (2014) 36–42

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Fourier–Mukai partners of singular genus one curves

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ARTICLE INFO

Article history:
Received 30 September 2013
Received in revised form 29 April 2014
Accepted 6 May 2014
Available online 20 May 2014

MSC:
primary 18E30
14F05
secondary 18E25
14H52

Keywords:
Fourier–Mukai partners
Elliptic curve
Autoequivalence
Kodaira degenerations

ABSTRACT

The objective of the paper is to prove that, as it happens for smooth elliptic curves, any Fourier–Mukai partner of a projective reduced Gorenstein curve of genus one and trivial dualizing sheaf, is isomorphic to itself.

1. Introduction

Let X be a projective scheme and $D^b_c(X)$ the bounded derived category of coherent sheaves on X. It is an interesting problem to find all projective schemes Y with $D^b_c(X) \simeq D^b_c(Y)$, that is, to determine the set $FM(X)$ of projective Fourier–Mukai partners of X.

The aim of this paper is to prove the following

Theorem 1.1. *Let X be a projective reduced connected Gorenstein curve of arithmetic genus one and trivial dualizing sheaf over an algebraically closed field k of characteristic zero. Then any projective Fourier–Mukai partner Y of X is isomorphic to X, that is, $FM(X) = \{X\}$.*

The curves appearing in the statement of the theorem were classified by Catanese in [1]. Namely, if X is a projective reduced connected Gorenstein curve of arithmetic genus one and trivial dualizing sheaf, by Proposition 1.18 in [1], it is isomorphic:

(1) either to a Kodaira curve (always with locally planar singularities), that is,
  (1.1) a smooth elliptic curve;
  (1.2) a rational curve with one node (following Kodaira's notation, that is a curve of type I_1);

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http://dx.doi.org/10.1016/j.geomphys.2014.05.011
0393-0440/

(1.3) a rational curve with one cusp (a curve of type I_2);
(1.4) a cycle of N rational smooth curves (a curve of type I_N) with $N \geq 2$;
(1.5) two rational smooth curves forming a tacnode curve (a curve of type II); or
(1.6) three concurrent rational smooth curves in the plane (a curve of type IV);

(2) or to a curve consisting of $N \geq 4$ rational smooth curves meeting at a point x where the tangents to the branches are linearly dependent, but any $(N-1)$ of them are independent.

Note that, by results of Kodaira and Miranda, the curves in (1) are exactly all the possible reduced fibers appearing in a smooth elliptic surface or in a smooth elliptic threefold. This explains why they are called Kodaira curves.

The theorem was just known for smooth elliptic curves. In this case, it was proved by Hille and Van den Bergh in [2]. For the integral singular curves in the above list, that is, for X a rational curve with one node or a cusp, Burban and Kreußler study in [3] the derived category $D^b_c(X)$ and its group $\mathrm{Aut}(D^b_c(X))$ of autoequivalences, but they do not tackle the question of Fourier–Mukai partners. Thus our contribution is to pass from the classical case of a smooth elliptic curve to the singular case generalizing the result to all singular curves of Catanese's list.

In 1998, Bridgeland computes all Fourier–Mukai partners of a smooth elliptic surface. He proves in [4] that the partners of relatively minimal smooth elliptic surfaces are certain relative compactified Jacobians. Some recent works [5,6] are concerned about higher dimensional elliptic fibrations. But, for the moment there is not a similar classification for the partners of higher dimensional varieties fibered by elliptic curves. Beyond its own interest, let us mention that one of the main applications of our Theorem is that it could be useful for classifying projective Fourier–Mukai partners of any elliptic fibration, that is, a flat morphism $p: Z \to S$ of projective schemes whose general fiber is a smooth elliptic curve. No more condition is required on either S or the total space Z that could be even singular. The reason is that Proposition 2.15 in [7] proves that a relative integral functor between the derived categories of two projective fibrations is an equivalence if and only if its restriction to every fiber (not only the smooth ones) is an equivalence. Then, if $q: W \to S$ is a relative Fourier–Mukai partner of $p: Z \to S$, for every $s \in S$ the fibers Z_s and W_s are Fourier–Mukai partners as well. Thus for example, by Miranda's result, to classify the partners of an elliptic threefold, it would be helpful to know about the partners of all the curves of (1) in Catanese's list. On the other hand, the structure of some relative compactified Jacobians for certain elliptic fibrations is already known (see [8]).

To finish, one should point out that elliptic fibrations have been used is string theory, notably in connection with mirror symmetry in Calabi–Yau manifolds and D-branes (see [9] for a good survey). Some of the classic examples of families of Calabi–Yau manifolds for which there is a description of the mirror family are elliptic fibrations [10]. Moreover, there is a relative Fourier–Mukai transform for most elliptic fibrations [11,7] that can be understood in terms of duality in string theory [12–14] or D-brane theory. More generally, due to the interpretation of B-type D-branes as objects of the derived category [15] and to Kontsevich's homological mirror symmetry proposal [15], one expects the Fourier–Mukai transform (or its relative version) to act on the spectrum of D-branes. The study of D-branes on Calabi–Yau manifolds inspired in fact the search of new Fourier–Mukai partners [16–19] among other mathematical problems.

In this paper, we will always work over an algebraically closed field k of characteristic zero and all schemes are assumed to be separated noetherian and of finite type over k.

2. The proof of Theorem 1.1

In the proof of our result, we will make use of the following facts.

2.1

Let X and Y be two proper schemes over k and let $\mathcal{K}^\bullet$ be an object in $D^-(X \times Y)$. The *integral functor* defined by $\mathcal{K}^\bullet$ is the functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y}: D^-(X) \to D^-(Y)$ given by

$$\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y_*}(\pi_X^* \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet),$$

where $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ are the two projections. The complex $\mathcal{K}^\bullet$ is said to be the kernel of the integral functor.

2.2

Our theorem is just stated for projective curves and the reason for it is that the projective context provides the following important advantage. Due to a famous result of Orlov [16], as extended by Ballard [20] to the singular case, if X and Y are projective schemes over k and $F: D^b_c(X) \simeq D^b_c(Y)$ is an equivalence of categories, then there is a unique (up to isomorphism) $\mathcal{K}^\bullet \in D^b_c(X \times Y)$ such that $F \simeq \Phi^{\mathcal{K}^\bullet}_{X \to Y}$, that is, any equivalence between the bounded derived categories of two projective schemes is a Fourier–Mukai functor. This result is now a consequence of a more general result by Lunts and Orlov [21].

2.3

The existence of a Fourier–Mukai functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y}: D^b_c(X) \simeq D^b_c(Y)$ between the derived categories of two projective schemes X and Y has important geometrical consequences. For instance, it is well known that if X is a smooth scheme,