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Planar symmetric concave central configurations in Newtonian four-body problems

Chunhua Deng [a,*], Shiqing Zhang [b]

[a] Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China

[b] College of Mathematics, Sichuan University, Chengdu 610064, China

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ABSTRACT

In this paper, we consider the problem: given a symmetric concave configuration of four bodies, under what conditions is it possible to choose positive masses which make it central. We show that there are some regions in which no central configuration is possible for positive masses. Conversely, for any configuration in the complement of the union of these regions, it is always possible to choose positive masses to make the configuration be a central configuration.

1. Introduction and main results

The Newtonian n-body problems [1–27] concern the motion of n points with masses $m_i \in \mathbb{R}^+$, $i = 1, 2, \ldots, n$. The motion is governed by Newton's second law and gravitational law:

$$m_i\ddot{q}_i = \sum_{k\neq i} \frac{m_k m_i (q_k - q_i)}{|q_k - q_i|^3}, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $q_i \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is the position of m_i. Alternatively the system (1.1) can be written

$$m_i\ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad i = 1, 2, \ldots, n \tag{1.2}$$

where

$$U(q) = U(q_1, q_2, \ldots, q_n) = \sum_{1\leq k<j\leq n} \frac{m_k m_j}{|q_k - q_j|} \tag{1.3}$$

is the Newtonian potential of system (1.1). Let

$$C = m_1q_1 + \cdots + m_nq_n, \qquad M = m_1 + \cdots + m_n, \qquad c = C/M$$

be the first momentum, total mass and center of masses, respectively. The set $\triangle$ of collision configurations is defined by

$$\triangle = \{q \in (\mathbb{R}^d)^n : q_i = q_j \text{ for some } i \neq j\}.$$

* Corresponding author.
E-mail address: chdeng8011@sohu.com (C. Deng).

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A configuration $q = (q_1, \ldots, q_n) \in (\mathbb{R}^d)^n \setminus \triangle$ is called a central configuration [23] if there exists some positive constant λ such that

$$-\lambda(q_i - c) = \sum_{j=1, j\neq i}^{n} \frac{m_j(q_j - q_i)}{|q_j - q_i|^3}, \quad i = 1, 2, \ldots, n. \tag{1.4}$$

Furthermore, it can be easily verified [15,23] that $\lambda = U/I$, where I is the moment of inertial of the system, i.e. $I = \sum_{i=1}^{n} m_i |q_i - c|^2$. The set of central configurations are invariant under three classes of transformations on $(\mathbb{R}^d)^n$: translations, scalings, and orthogonal transformations.

Several aspects of the n-body problem motivate the study of central configurations, one can see [15–19,21–23], which allow the computation of homographic solutions, and if the n bodies are heading for a simultaneous collision, then the bodies tend to a central configuration; central configurations also appear as a key point when studying the topological changes of the integral manifolds.

Some results of central configurations are given in [1–26] for N-body problems with given masses. Specially, Hampton–Moeckel [8] proved the finiteness of central configurations (up to symmetry) for 4-body problems with any given positive masses, they prove the numbers of central configurations (up to symmetry) are between 32 and 8472, although it is hard to see the shapes of central configurations except the collinear case, but this is still an important progress about Smale–Wintner's finiteness conjecture of central configurations (up to symmetry) for n-body problems with any given positive masses [22,23]. Cors–Roberts [5] studied four-body co-circular central configurations, they classify the set of central configurations lying on a common circle in the Newtonian 4-body problem with positive masses, they proved the set is a two-dimensional surface, specially, they proved that if any two masses of a four-body co-circular central configuration are equal, then the configuration has a line of symmetry, but there are 4-body central configurations with different masses and without symmetries, in fact, we can find such central configurations in the paper of Cors–Roberts [5].

A configuration $q = (q_1, \ldots, q_4)$ is concave if one mass point is in the interior of the triangle formed by the other three mass points. Long and Sun [11] proved:

Lemma 1.1. *Let $\alpha, \beta > 0$ be any two given real numbers. Let $q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$ be a concave non-collinear central configuration with masses $(\beta, \alpha, \beta, \beta)$ respectively, and with q_2 located inside the triangle formed by q_1, q_3, and q_4. Then the configuration q must possess a symmetry, so either q_1, q_3, and q_4 form an equilateral triangle and q_2 is located at the center of the triangle, or q_1, q_3, and q_4 form an isosceles triangle, and q_2 is on the symmetrical axis of the triangle.*

It is natural to consider the inverse problem: given a configuration, find mass vectors such that the configuration is a central configuration. Moulton [16], Wintner [23] considered the inverse problem for collinear n-body problem. Albouy and Moeckel [1] proved that a given configuration determines a two-parameter family of masses making it central where masses are allowed to be negative. Ouyang and Xie [24] found the region for collinear 4-body configurations for which there exists a positive mass vector making it central. For a given configuration, it can be a central configuration for infinitely many mass vectors. In particular, a configuration is called super central configuration if it is a central configuration for a mass vector m and it is also a central configuration for another mass vector m' where m' is a permutation of m and $m \neq m'$. Xie [26] proved the existence of the super central configurations in the collinear four-body problem and the results were used to study the number of central configurations under geometric equivalence in [27].

Marshall Hampton gave some results (Theorem 6, 7 and 8 in [6]) of concave central configurations for four-body problem. They found that for the outer triangle with suitable parameter, there is a unique concave central configuration for which the fourth point traces out a simple curve. In this paper we consider the inverse problem: given a planar symmetric concave configuration (Fig. 1), find the positive mass vectors, if any, for which it is a central configuration.

The equations for the central configurations can be written as

$$\begin{cases} m_2 \dfrac{q_2 - q_1}{|q_2 - q_1|^3} + m_3 \dfrac{q_3 - q_1}{|q_3 - q_1|^3} + m_4 \dfrac{q_4 - q_1}{|q_4 - q_1|^3} = -\lambda(q_1 - c) \\ m_1 \dfrac{q_1 - q_2}{|q_1 - q_2|^3} + m_3 \dfrac{q_3 - q_2}{|q_3 - q_2|^3} + m_4 \dfrac{q_4 - q_2}{|q_4 - q_2|^3} = -\lambda(q_2 - c) \\ m_1 \dfrac{q_1 - q_3}{|q_1 - q_3|^3} + m_2 \dfrac{q_2 - q_3}{|q_2 - q_3|^3} + m_4 \dfrac{q_4 - q_3}{|q_4 - q_3|^3} = -\lambda(q_3 - c) \\ m_1 \dfrac{q_1 - q_4}{|q_1 - q_4|^3} + m_2 \dfrac{q_2 - q_4}{|q_2 - q_4|^3} + m_3 \dfrac{q_3 - q_4}{|q_3 - q_4|^3} = -\lambda(q_4 - c). \end{cases} \tag{1.5}$$

We can obtain the following results:

Theorem 1.1. *Let $q_1 = (-1, 0)$, $q_2 = (1, 0)$, $q_3 = (0, t)$, $q_4 = (0, s)$ where $t > s > 0$, and assume that the center of mass $c = C/M = (c_x, c_y) = q_4$. The symmetric concave configuration $q = (q_1, q_2, q_3, q_4)$ can be a central configuration if and only*