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Noncommutative cross-ratios

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ABSTRACT

We present a definition and discuss basic properties of cross-ratios over noncommutative skew-fields.

1. Introduction

The goal of this note is to present a definition and to discuss basic properties of cross-ratios over (noncommutative) division rings or skew-fields. We present the noncommutative cross-ratios as products of quasi-Plücker coordinates introduced in [1] (see also [2]). Actually, noncommutative cross-ratios were already mentioned in a remark in [1]. I decided to return to the subject after my colleague Feng Luo explained to me the importance of cross-ratios in modern geometry (see, for example, [3–5]).

2. Quasi-Plücker coordinates

We recall here only the theory of quasi-Plücker coordinates for $2 \times n$-matrices over a noncommutative division ring $\mathcal{F}$. For general $k \times n$-matrices the theory is presented in [1,2]. Recall (see [6,7] and subsequent papers) that for a matrix $\begin{pmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{pmatrix}$ one can define four quasideterminants provided the corresponding elements are invertible:

$$\begin{vmatrix} \boxed{a_{1k}} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix} = a_{1k} - a_{1i}a_{2i}^{-1}a_{2k}, \qquad \begin{vmatrix} a_{1k} & \boxed{a_{1i}} \\ a_{2k} & a_{2i} \end{vmatrix} = a_{1i} - a_{1k}a_{2k}^{-1}a_{2i},$$

$$\begin{vmatrix} a_{1k} & a_{1i} \\ \boxed{a_{2k}} & a_{2i} \end{vmatrix} = a_{2k} - a_{2i}a_{1i}^{-1}a_{1k}, \qquad \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & \boxed{a_{2i}} \end{vmatrix} = a_{2i} - a_{2k}a_{1k}^{-1}a_{1i}.$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix}$ be a matrix over $\mathcal{F}$.

Lemma 2.1. *Let $i \neq k$. Then*

$$\begin{vmatrix} a_{1k} & \boxed{a_{1i}} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & \boxed{a_{1j}} \\ a_{2k} & a_{2j} \end{vmatrix} = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & \boxed{a_{2i}} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & \boxed{a_{2j}} \end{vmatrix}^{-1}$$

if the corresponding expressions are defined.

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Note that in the formula the boxed elements on the left and on the right must be on the same level.

Definition 2.2. We call the expressions

$$q_{ij}^k(A) = \begin{vmatrix} a_{1k} & \boxed{a_{1i}} \\ a_{2k} & a_{2i} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & \boxed{a_{1j}} \\ a_{2k} & a_{2j} \end{vmatrix} = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & \boxed{a_{2i}} \end{vmatrix}^{-1} \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & \boxed{a_{2j}} \end{vmatrix}^{-1}$$

the quasi-Plücker coordinates of matrix A.

Our terminology is justified by the following observation. Recall that in the commutative case the expressions

$$p_{ik}(A) = \begin{vmatrix} a_{1i} & a_{1k} \\ a_{2i} & a_{2k} \end{vmatrix} = a_{1i}a_{2k} - a_{1k}a_{2i}$$

are the Plücker coordinates of A. One can see that in the commutative case

$$q_{ij}^k(A) = \frac{p_{jk}(A)}{p_{ik}(A)},$$

i.e. quasi-Plücker coordinates are ratios of Plücker coordinates.

Let us list properties of quasi-Plücker coordinates over (noncommutative) division ring $\mathcal{F}$. We sometimes write q_{ij}^k instead of $q_{ij}^k(A)$ where it cannot lead to a confusion.

(1) Let g be an invertible 2 by 2 matrix over $\mathcal{F}$. Then

$$q_{ij}^k(g \cdot A) = q_{ij}^k(A).$$

(2) Let $\Lambda = \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ be an invertible diagonal matrix over $\mathcal{F}$. Then

$$q_{ij}^k(A \cdot \Lambda) = \lambda_i^{-1} \cdot q_{ij}^k(A) \cdot \lambda_j.$$

(3) If $j = k$ then $q_{ij}^k = 0$; if $j = i$ then $q_{ij}^k = 1$ (we always assume $i \neq k$).

(4) $q_{ij}^k \cdot q_{j\ell}^k = q_{i\ell}^k$. In particular, $q_{ij}^k q_{ji}^k = 1$.

(5) "Noncommutative skew-symmetry": For distinct i, j, k

$$q_{ij}^k \cdot q_{jk}^i \cdot q_{ki}^j = -1.$$

One can also rewrite this formula as $q_{ij}^k q_{jk}^i = -q_{ik}^j$.

(6) "Noncommutative Plücker identity": For distinct i, j, k, ℓ

$$q_{ij}^k q_{ji}^\ell + q_{i\ell}^k q_{\ell i}^j = 1.$$

One can easily check two last formulas in the commutative case. In fact,

$$q_{ij}^k \cdot q_{jk}^i \cdot q_{ki}^j = \frac{p_{jk}p_{ki}p_{ij}}{p_{ik}p_{ji}pkj} = -1$$

because Plücker coordinates are skew-symmetric: $p_{ij} = -p_{ji}$ for any i, j.

Also, assuming that $i < j < k < \ell$

$$q_{ij}^k q_{ji}^\ell + q_{i\ell}^k q_{\ell i}^j = \frac{p_{jk}p_{i\ell}}{p_{ik}p_{j\ell}} + \frac{p_{\ell k}p_{ij}}{p_{ik}p_{\ell j}}.$$

Because $\frac{p_{\ell k}}{p_{\ell j}} = \frac{p_{k\ell}}{p_{j\ell}}$, the last expression equals

$$\frac{p_{jk}p_{i\ell}}{p_{ik}p_{j\ell}} + \frac{p_{k\ell}p_{ij}}{p_{ik}pj\ell} = \frac{p_{ij}p_{k\ell} + p_{i\ell}p_{jk}}{p_{ik}p_{j\ell}} = 1$$

due to the celebrated Plücker identity

$$p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk} = 0.$$

Remark 2.3. We presented here a theory of the *left* quasi-Plücker coordinates for 2 by n matrices where $n > 2$. A theory of the *right* quasi-Plücker coordinates for n by 2 or, more generally, for n by k matrices where $n > k$ can be found in [1,2].

3. Definition and basic properties of cross-ratios

We define cross-ratios over (noncommutative) division ring $\mathcal{F}$ by imitating the definition of classical cross-ratios in homogeneous coordinates, namely, if the four points are represented in homogeneous coordinates by vectors a, b, c, d such that $c = a + b$ and $d = ka + b$, then their cross-ratio is k.