Journal of Geometry and Physics 82 (2014) 64–74

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On the local structure of noncommutative deformations

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ARTICLE INFO

Article history:
Received 14 November 2013
Received in revised form 18 March 2014
Accepted 6 April 2014
Available online 13 April 2014

MSC:
primary 58B34
53D17

Keywords:
Contravariant connections
Metacurvature
Noncommutative deformations

ABSTRACT

Let $(M, \pi, \mathcal{D})$ be a Poisson manifold endowed with a flat, torsion-free contravariant connection. We show that if $\mathcal{D}$ is an $\mathcal{F}$-connection then there exists a tensor $\mathbf{T}$ such that $\mathcal{D}\mathbf{T}$ is the metacurvature tensor introduced by E. Hawkins in his work on noncommutative deformations. We compute $\mathbf{T}$ and the metacurvature tensor in this case and show that if $\mathbf{T} = 0$ then near any regular point π and $\mathcal{D}$ are defined in a natural way by a Lie algebra action and a solution of the classical Yang–Baxter equation. Moreover, when $\mathcal{D}$ is the contravariant Levi-Civita connection associated to π and a Riemannian metric, the Lie algebra action can be chosen in such a way that it preserves the metric. This solves the inverse problem of a result of the second author.

1. Introduction and main result

In [1,2] Hawkins showed that if a deformation of the graded algebra $\Omega^*(M)$ of differential forms on a Riemannian manifold M comes from a spectral triple describing M, then the Poisson tensor π (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

(H_1) the associated metric contravariant connection $\mathcal{D}$ is flat;
(H_2) the metacurvature of $\mathcal{D}$ vanishes;
(H_3) π is compatible with the Riemannian volume μ, i.e., $d(i_\pi \mu) = 0$.

The metric contravariant connection associated naturally to any pair of a (pseudo-)Riemannian metric and a Poisson tensor is the contravariant analogue of the classical Levi-Civita connection; it has appeared first in [3]. The metacurvature, introduced in [2], is a $(2, 3)$-type tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any flat, torsion-free contravariant connection.

The main result of Hawkins [2, Theorem 6.6 and also Lemma 6.5] states that if (M, π, g) is a triple satisfying (H_1)–(H_3) with M compact, then around any regular point $x_0 \in M$ the Poisson tensor can be written as

$$\pi = \sum_{i,j} a^{ij} X_i \wedge X_j \tag{1}$$

where the matrix (a^{ij}) is constant and invertible and $\{X_1, \ldots, X_s\}$ is a family of linearly independent commuting Killing vector fields.

On the other hand, the second author showed in [4] that if $\zeta : \mathfrak{g} \to \mathfrak{X}^1(M)$ is an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M and $r \in \wedge^2 \mathfrak{g}$ is a solution of the classical Yang–Baxter equation, then:

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http://dx.doi.org/10.1016/j.geomphys.2014.04.003
0393-0440/

(a) The map $\mathcal{D}^r : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ given by

$$\mathcal{D}^r_\alpha \beta := \sum_{i,j=1}^{n} a^{ij} \alpha(\zeta(u_i)) \mathcal{L}_{\zeta(u_j)} \beta, \tag{2}$$

where $\{u_1, \dots, u_n\}$ is any basis of $\mathfrak{g}$ and a^{ij} are the components of r in this basis, depends only on r and ζ and defines a flat, torsion-free contravariant connection with respect to the Poisson tensor $\pi^r := \zeta(r)$.

(b) If M is Riemannian and ζ preserves the metric, $\mathcal{D}^r$ is nothing else but the metric contravariant connection associated to the metric and π^r.

(c) If $\mathfrak{g}$ acts freely on M, the metacurvature of $\mathcal{D}^r$ vanishes.

In this setting, (1) can be re-expressed by saying that there exists a free action $\zeta : \mathfrak{g} \to \mathfrak{X}^1(U)$ of a finite-dimensional abelian Lie algebra $\mathfrak{g}$ on an open neighborhood $U \subseteq M$ of x_0 which preserves g, and a solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation such that $\pi = \pi^r$. Moreover, since ζ preserves g, then $\mathcal{D} = \mathcal{D}^r$ by (b). It follows that $\mathcal{D}$ is a Poisson connection, i.e., $\mathcal{D}\pi = 0$ and hence an $\mathcal{F}^{\text{reg}}$-connection (see [5]).

Given a flat, torsion-free $\mathcal{F}^{\text{reg}}$-connection $\mathcal{D}$ on a Poisson manifold (M, π), we shall see that there exists a (2, 2)-type tensor field $\mathbf{T}$ on the dense open set of regular points such that

(i) $\mathcal{D}\mathbf{T} = \mathcal{M}$ where $\mathcal{M}$ is the metacurvature of $\mathcal{D}$;

(ii) $\mathbf{T}$ vanishes if and only if the exterior differential of any parallel 1-form is also parallel.

By looking at the proof of the second author's result closely, one observes that in proving (c) the second author showed that $\mathcal{D}^r$ is an $\mathcal{F}^{\text{reg}}$-connection and that whenever a 1-form is $\mathcal{D}^r$-parallel then so is its exterior differential, meaning that $\mathbf{T}$ vanishes. Accordingly, (c) can be rephrased as follows:

(c′) If $\mathfrak{g}$ acts freely on M, $\mathcal{D}^r$ is an $\mathcal{F}^{\text{reg}}$-connection and $\mathbf{T}$ vanishes (and hence so does $\mathcal{M}$).

Note that in the case studied by Hawkins $\mathbf{T}$ vanishes since as we saw above the action ζ is free. So it is natural to consider the following problem, inverse of the second author's result: *Given a smooth manifold M endowed with a Poisson tensor π and a Riemannian metric g such that the associated metric contravariant connection is a flat $\mathcal{F}^{\text{reg}}$-connection and such that $\mathbf{T} = 0$, is there a free action of a finite-dimensional Lie algebra $\mathfrak{g}$ preserving g and a solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation such that $\pi = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$?*

The main result of this paper answers in the affirmative to that question in a more general setting. More precisely,

Theorem 1.1. *Let $(M, \pi, \mathcal{D})$ be a Poisson manifold endowed with a flat, torsion-free contravariant connection.*

(1) *If $\mathcal{D}$ is an $\mathcal{F}^{\text{reg}}$-connection and $\mathbf{T} = 0$, then for any regular point x_0 with rank $2r$, there exists a free action $\zeta : \mathfrak{g} \to \mathfrak{X}(U)$ of a $2r$-dimensional real Lie algebra $\mathfrak{g}$ on a neighborhood U of x_0, and an invertible solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation, such that $\pi = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$.*

(2) *Moreover, if $\mathcal{D}$ is the metric contravariant connection associated to π and a Riemannian metric g, then the action can be chosen in such a way that its fundamental vector fields are Killing.*

The paper is organized as follows. In Section 2, we recall some standard facts about contravariant connections and the metacurvature tensor; we also define the tensor $\mathbf{T}$. Section 3 is devoted to the computation of the metacurvature tensor (and the tensor $\mathbf{T}$ as well) in the case of an $\mathcal{F}^{\text{reg}}$-connection. In Section 4, we give a proof of Theorem 1.1.

Notation 1.2. For a smooth manifold M, $\mathcal{C}^\infty(M)$ will denote the space of smooth functions on M, $\Gamma(V)$ will denote the space of smooth sections of a vector bundle V over M, $\Omega^p(M) := \Gamma(\wedge^p T^*M)$ will denote the space of differential p-forms, and $\mathfrak{X}^p(M) := \Gamma(\wedge^p TM)$ will denote the space of p-vector fields.

For a Poisson tensor π on M, we will denote by $\pi_\sharp : T^*M \to TM$ the anchor map defined by $\beta(\pi_\sharp(\alpha)) = \pi(\alpha, \beta)$, and by H_f the Hamiltonian vector field of a function f, that is, $H_f := \pi_\sharp(df)$. We will also denote by $[\,,\,]_\pi$ the Koszul–Schouten bracket on differential forms (see, e.g., [6]); this is given on 1-forms by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\sharp(\alpha)}\beta - \mathcal{L}_{\pi_\sharp(\beta)}\alpha - d\big(\pi(\alpha, \beta)\big).$$

The symplectic foliation of (M, π) will be denoted by $\mathcal{S}$, and $T\mathcal{S} = \operatorname{Im} \pi_\sharp$ will be its associated tangent distribution. Finally, we will denote by M^{reg} the dense open set where the rank of π is locally constant.

2. Preliminaries

2.1. Contravariant connections

Contravariant connections on Poisson manifolds were defined by Vaismann [7] and studied in detail by Fernandes [8]. These connections play an important role in Poisson geometry (see for instance [8,9]) and have recently turned out to be useful in other branches of mathematics (e.g., [1,2]).