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Instantons on the exceptional holonomy manifolds of Bryant and Salamon

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ABSTRACT

We give a construction of G_2 and $Spin(7)$ instantons on exceptional holonomy manifolds constructed by Bryant and Salamon, by using an ansatz of spherical symmetry coming from the manifolds being the total spaces of rank-4 vector bundles. In the G_2 case, we show that, in the asymptotically conical model, the connections are asymptotic to Hermitian Yang–Mills connections on the nearly Kähler $S^3 \times S^3$.

1. Introduction

A natural situation in which to study gauge fields is when the Riemannian manifold (M, g) has a special holonomy group, in which case the holonomy group determines a subalgebra $\mathfrak{g} \subseteq \Lambda^2$. For G_2, $Spin(7)$ and Calabi–Yau manifolds, if the curvature F_A of a connection lies in $\mathfrak{g}$, the connection automatically satisfies the Yang–Mills equations.

Following the constructions of Bryant and Salamon, and of Joyce, the study of gauge fields on reduced holonomy manifolds was formulated in the physics literature (see for example [1,2]) and also in mathematics (see [3,4]). Existence theorems have been given on both compact and non-compact spaces in, for example, [5–9].

Here we consider the problem of constructing solutions to the instanton equation on the non-flat manifolds of reduced holonomy that were obtained by Bryant and Salamon. In [10] a number of complete metrics of holonomy exactly G_2 and $Spin(7)$ were constructed. Their work stemmed from a principal bundle construction to obtain a family of non-degenerate differential forms that depend on 2 functions of one variable. In each given case they were able to solve a system of ordinary differential equations so as to obtain a metric g_γ of exceptional holonomy.

The differentiable manifolds that support the metrics are the total spaces of rank 3 and 4 vector bundles over 3 and 4-dimensional manifolds. In one example for the group G_2, the manifold X is given as the total space of the spinor bundle $\mathcal{S} \to S^3$ over the three-sphere. The metric g_γ is then given by

$$g_\gamma = 4(1+r)^{-1/3} d\sigma^2 + 3(1+r)^{2/3} ds^2$$

where ds^2 is the pullback of the round metric tensor on S^3 and $d\sigma^2$ is a quadratic form on X that restricts to the fibres of the spinor bundle to be the standard flat metric. That is, g_γ restricts to the fibres to be conformally equivalent to the flat metric on $\mathbb{R}^4$.

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The original ansatz used to obtain instanton solutions to the Yang–Mills equations in 4-dimensions was to use the assumption of spherical symmetry, and to construct

$$A = \frac{\mathrm{Im}(x d\bar{x})}{1 + |x|^2} \tag{1.1}$$

so that $\nabla = d + A$ is an $SU(2)$-connection with self-dual curvature on $\mathbb{R}^4$ (see [11]).

In the example at hand, we take the $\mathfrak{su}(2)$-valued form

$$A_1 = \mathrm{Im}(a\bar{\alpha})$$

and construct a connection whose curvature lies in the subspace $\mathfrak{g}_2 \otimes \mathfrak{su}(2) \subseteq \Lambda^2 \otimes \mathfrak{su}(2)$. Neither the quaternion-valued function a or form α are well defined on X, but the form A_1 is well-defined. We set $r = |a|^2 = a\bar{a}$ to be the squared fibre radius function on $\mathcal{S}$. This function is well-defined on X.

Theorem 1.1. *Let X be the total space of the spinor bundle over the round S^3 of constant curvature κ, equipped with the G_2-holonomy metric that is determined by the form γ constructed by Bryant and Salamon. For any $C > -3$ let $f(r)$ be the function*

$$f(r) = \frac{2}{3(r+1) + C(r+1)^{1/3}}.$$

Then the connection $\nabla = d + f(r)A_1$ defines a non-trivial G_2-instanton on the trivial rank 2 complex vector bundle over X. The connections are invariant under the group $SU(2)^3$ of isometries of X.

The first statements of this result are proven in Section 4. The statement on the invariance is shown in Section 5.

It has also been noted that the metrics of Bryant and Salamon are asymptotically conical. That is, outside of sufficiently large compact sets, the metric g_γ is arbitrarily close to a conical metric

$$d\rho^2 + \rho^2 g$$

defined on $\mathbb{R}^+ \times (S^3 \times S^3)$. The closeness is measured with respect to the conical metric. A consequence of (X, g_γ) having G_2 holonomy is that $(S^3 \times S^3, g)$ must be *nearly Kähler*. This is to say that it admits an almost-complex structure compatible with the metric and a complex (3, 0)-form that satisfies certain differential relations with the Hermitian form of the metric. In this context we can consider the Hermitian Yang–Mills equations for connections on principal and vector bundles over $S^3 \times S^3$. We have the following result on the asymptotic behaviour of the connections constructed in Theorem 1.1.

Theorem 1.2. *There exists a non-flat Hermitian Yang–Mills connection $\widetilde{A}$ on the trivial $\mathbb{C}^2$ bundle over $S^3 \times S^3$ such that every G_2-instanton on X constructed in Theorem 1.1 is asymptotic to the pull-back of $\widetilde{A}$ to $\mathbb{R}^+ \times (S^3 \times S^3)$.*

This result is demonstrated in Section 5.

To the author's knowledge, this example was previously unknown, though it would be interesting to understand it in comparison with the works of Bryant, and Harland and Nölle [12,13].

The results in this paper have much in common with the recent work of Oliveira [14]. The principal difference is that he considers monopoles and instantons on the other G_2 metrics of Bryant and Salamon, those being total spaces of rank-3 vector bundles over 4-manifolds. The convenient model for the metric in that case is that of [15], rather than the original model of Bryant and Salamon.

The Bryant–Salamon construction for metrics of $Spin(7)$-holonomy involves considering the negative spinor bundle on S^4 and constructing a non-degenerate 4-form. Bryant and Salamon solved a system of ODEs to obtain an example for which the form Ψ is closed. We are able to construct a $Spin(7)$-instanton with structure group $SU(2)$ for this $Spin(7)$ structure.

Let $\mathcal{S}^- \to S^4$ be the negative spinor bundle on S^4 and denote by Y^8 the total space of this manifold. From the bundle $\mathcal{S}^-$, we can construct a principal $SU(2)$-bundle $\mathcal{E}_Y$ over Y and consider connection forms on $\mathcal{E}_Y$ of the form

$$A = \phi + f(r)A_2$$

where ϕ is pulled back from the connection on $\mathcal{S}^-$ for the round metric on S^4, and $A_2 = \mathrm{Im}(\bar{a}\alpha)$ is an equivariant 1-form on $\mathcal{E}_Y$. $f(r)$ is a function on X that depends only on the radius in the $\mathcal{S}^-$-fibre directions.

Theorem 1.3. *Let Y be the total space of the negative spinor bundle on S^4, equipped with the $Spin(7)$-holonomy metric of Bryant and Salamon. Let $\mathcal{E}_Y$ be the associated $SU(2)$-bundle on Y and let $f(r)$ be the function*

$$f(r) = \frac{1}{r(1 + D(1+r)^{3/5})} + \frac{D(2r+5)}{5r(1+r)^{2/5}(1 + D(1+r)^{3/5})}$$

where $D > -1$ is a constant. Then the connection $A = \phi + fA_2$ defines a non-trivial $Spin(7)$-instanton on $\mathcal{E}_Y$ with structure group $SU(2)$.

The connection form A is defined only away from the set $\{r = 0\}$ in Y, that being the image of the zero section of $\mathcal{S}^-$. This theorem is proved in Section 6.