Journal of Geometry and Physics 78 (2014) 1–11

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

There exist no 4-dimensional geodesically equivalent metrics with the same stress–energy tensor

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ARTICLE INFO

Article history:
Received 21 February 2013
Received in revised form 3 January 2014
Accepted 4 January 2014
Available online 10 January 2014

Keywords:
Geodesically equivalent metrics
Einstein field equation
Warped product metrics

ABSTRACT

We show that if two 4-dimensional metrics of arbitrary signature on one manifold are geodesically equivalent (i.e., have the same geodesics considered as unparameterized curves) and are solutions of the Einstein field equation with the same stress–energy tensor, then they are affinely equivalent or flat. If we additionally assume that the metrics are complete or that the manifold is closed, the result remains valid in all dimensions ≥ 3.

1. Definitions and results

Let (M^n, g) be a connected pseudo-Riemannian manifold of arbitrary signature of dimension $n \geq 3$.

We say that a metric $\bar g$ on M^n is *geodesically equivalent* to g, if every geodesic of g is a (possibly, reparametrized) geodesic of $\bar g$. We say that $\bar g$ is *affinely equivalent* to g, if the Levi-Civita connections of g and $\bar g$ coincide.

In this paper we study the question whether two geodesically equivalent metrics g and $\bar g$ can satisfy the Einstein field equation with the same stress–energy tensor:

$$R_{ij} - \frac{R}{2} \cdot g_{ij} = \bar R_{ij} - \frac{\bar R}{2} \cdot \bar g_{ij}, \tag{1}$$

where R_{ij} ($\bar R_{ij}$, respectively) is the Ricci tensor of the metric g ($\bar g$, respectively), and $R := R_{ij} g^{ij}$ ($\bar R := \bar R_{ij} \bar g^{ij}$, respectively, $\bar g^{k\ell}$ is the tensor dual to $\bar g_{ij}$: $\bar g^{si} \bar g_{sj} = \delta^i{}_j$) is the scalar curvature.

There exist the following trivial examples of such a situation:

1. If geodesically equivalent metrics g and $\bar g$ are flat, then their stress–energy tensors vanish identically and therefore coincide. Examples of geodesically equivalent flat metrics are classically known and can be constructed as follows: take the classical projective transformation p of $(U \subseteq \mathbb{R}^n, g_{\text{standard}})$ (i.e., a local diffeomorphism that takes straight lines to straight lines, there is a $(n^2 + 2n)$-dimensional group of it) and consider the pullback of the standard euclidean metric g_{standard}; $\bar g = p^* g_{\text{standard}}$. It is clearly flat and geodesically equivalent to the initial metric g_{standard}. If p is not a classical affine transformation (the subgroup of affine transformations is $n^2 + n$-dimensional), $\bar g$ is not affinely equivalent to g_{standard}.
2. If g and $\bar g$ are affinely equivalent metrics with vanishing scalar curvature, then their stress–energy tensors coincide with the Ricci tensors and therefore coincide (since even Riemannian curvature tensors coincide). There are many examples of such a situation, a possibly simplest one is as follows: Take an arbitrary metric $h = h_{ij},\ i, j = 2, \ldots, n$ of zero scalar curvature on $\mathbb{R}^{n-1}(x_2, \ldots, x_n)$ and consider the direct product metric $g = dx_1^2 + \sum_{i,j=2}^n h_{ij} dx_i dx_j$ on

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0393-0440/$ – see front matter
http://dx.doi.org/10.1016/j.geomphys.2014.01.002

$\mathbb{R}^n = \mathbb{R}(x_1) \times \mathbb{R}^{n-1}(x_2, \ldots, x_n)$. Then, for this metric, and also for the (affinely equivalent) metric $g = dx_1^2 + 2\sum_{i,j=2}^n h_{ij}dx_idx_j$, the scalar curvature is zero.

3. The metric $\bar{g} := \text{const} \cdot g$ has the same stress–energy tensor as g. Indeed, $R_{ij} = \bar{R}_{ij}$, and $\bar{R} := \bar{g}^{ij}R_{ij} = \frac{1}{\text{const}}R$ so $Rg_{ij} = \frac{1}{\text{const}}R \cdot \text{const}\, g_{ij} = \bar{R}\bar{g}_{ij}$.

In the present paper we show that in dimensions 3 and 4 this list of trivial examples contains all possibilities:

Theorem 1. *If two geodesically equivalent metrics g and $\bar{g}$ on a connected manifold M of dimension 3 or 4 satisfy (1), then at least one of the following possibilities takes place:*

1. *g and $\bar{g}$ are affinely equivalent metrics with zero scalar curvature, or*
2. *g and $\bar{g}$ are flat, or*
3. *$\bar{g} = const\ g$ for a certain $const \in \mathbb{R}$.*

By this theorem, unparameterized geodesics determine the Levi-Civita connection of a 3 or 4-dimensional metric uniquely within the solutions of the Einstein field equation with the same stress–energy tensor provided the metric is not flat.

The motivation to study this question came from physics. It is known that geodesics of a space–time metric correspond to the trajectories of the free falling uncharged particles, and that certain astronomical observations give the trajectories of free falling uncharged particles as unparameterized curves; moreover, unparameterized geodesics and how and whether they determine the metric were actively studied by theoretical physicists (cf. [1–4]) in the context of general relativity. The space–time metric is a solution of the Einstein equation (there of course could be many solutions of the Einstein equation with the same stress–energy tensor) and our theorem implies that if we know the (unparameterized) trajectories of free falling uncharged particles and the stress–energy tensor, then we know (i.e., can in theory reconstruct) the metric or at least the Levi-Civita connection of the metric.

The dimension 4 is probably the dimension that could be interesting for physics, since space–time metrics are naturally 4-dimensional. The result for dimension 3 is essentially easier; that is why we put it here. In dimension two, the stress–energy tensor of every metric is identically zero and (the analog of) Theorem 1 is evidently wrong. It is also wrong in higher dimensions, we show an example in dimensions ≥ 5. The metrics g and $\bar{g}$ in this example both have zero scalar curvature and their Riemannian curvature tensors coincide. We do not know whether all geodesically equivalent not affinely equivalent metrics with the same stress–energy tensors have zero scalar curvature, but can show that the scalar curvature must be constant.

Theorem 2. *Suppose two nonproportional geodesically equivalent metrics g and $\bar{g}$ on a connected manifold M^n of dimension $n \geq 5$ satisfy (1). Then, the scalar curvatures of the metrics are constant.*

Combining this theorem with [5,6], we obtain that in the global setting, when the manifold is closed (= compact without boundary), or when both metrics are complete, the analog of Theorem 1 is still true in all dimensions.

We say that a (complete in both directions) g-geodesic $\gamma : \mathbb{R} \to M$ is $\bar{g}$-complete, if there exists a diffeomorphism $\tau : \mathbb{R} \to \mathbb{R}$ such that the curve $\bar{\gamma} := \gamma \circ \tau$ is a $\bar{g}$-geodesic.

Corollary 1. *Let M^n be a connected manifold of dimension $n \geq 5$. Suppose g and $\bar{g}$ on M^n are geodesically equivalent and satisfy (1). Assume in addition that g has indefinite signature and that every light-like g-geodesic γ is complete in both directions and is $\bar{g}$-complete.*

Then, the metrics are affinely equivalent.

Corollary 2. *Suppose two geodesically complete geodesically equivalent metrics g and $\bar{g}$ on a connected M^n, $n \geq 5$, such that g is positively definite or negatively definite, satisfy (1). Then, the metrics are affinely equivalent.*

Corollary 3. *Suppose two geodesically equivalent metrics g and $\bar{g}$ on a closed connected M^n, $n \geq 5$, satisfy (1). Then, the metrics are affinely equivalent.*

Probably the most famous special case of Theorem 1 that was known before is due to A. Z. Petrov [2] (see also [7,8]): he showed that 4-*dimensional Ricci-flat nonflat metrics of Lorentz signature cannot be geodesically equivalent, unless they are affinely equivalent.* It is one of the results for which Petrov obtained in 1972 the Lenin prize, the most important scientific award of the Soviet Union.

Remark 1. We observe the same effect that was observed in [8]: as in [8], in dimension 4 we have a rigidity (in [8] it was proved that 4-dimensional Einstein metrics of nonconstant curvature do not allow nontrivial geodesic equivalence. In every dimension $n \geq 5$ there exist Einstein metrics of nonconstant curvature that are geodesically equivalent but not affinely equivalent). We do not have any conceptual explanation for this effect, and in our proof the dimension is used in many arguments.