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Gamma structures and Gauss's contiguity

Vasily Golyshev [a,*], Anton Mellit [b]

[a] *Algebra and Number Theory Sector, Institute for Information Transmission Problems, Bolshoy Karetny per. 19, Moscow 127994, Russia*

[b] *International Centre for Theoretical Physics, Strada Costiera, 11 I - 34151 Trieste, Italy*

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ABSTRACT

We introduce gamma structures on regular hypergeometric D-modules in dimension 1 as special one-parametric systems of solutions on the compact subtorus. We note that a balanced gamma product is in the Paley–Wiener class and show that the monodromy with respect to the gamma structure is expressed algebraically in terms of the hypergeometric exponents. We compute the hypergeometric monodromy explicitly in terms of certain diagonal matrices, Vandermonde matrices and their inverses (or generalizations of those in the resonant case).

A hypergeometric D-module with rational exponents is motivic, i.e. may be realized as a constituent of the pushforward of the constant D-module $\mathcal{O}$ in a pencil of varieties over $\mathbf{G_m}$ defined over $\overline{\mathbb{Q}}$. As a result, the vector space of solutions is endowed with two K-rational structures for some number field K that come from the Betti (resp. de Rham) structures on the cohomology of the fibers. On the other hand, no rational structure exists in the case of irrational exponents, and yet one still wishes to have the benefits of the Dwork/Boyarsky method of parametric exponents.

A substitute is the *gamma structure* on a hypergeometric D-module which manifests itself as a rational structure in the case of rational exponents.

We follow Katz's treatment [1] of hypergeometrics in order to fix our basics. Let $\mathbf{G_m} = \operatorname{Spec} \mathbb{C}[z, z^{-1}]$ be a one-dimensional torus. By $\mathcal{D}$ denote the algebra of differential operators on $\mathbf{G_m}$, by D denote the differential operator $z\frac{\partial}{\partial z}$. One has $\mathcal{D} = \mathbb{C}[z, z^{-1}, D]$.

Definition. Let n and m be a pair of nonnegative integers. Let P and Q be polynomials of degrees n and m respectively. Define *a hypergeometric differential operator* of type (n, m):

$$H(P, Q) = P(D) - zQ(D),$$

and *the hypergeometric D-module*:

$$\mathcal{H}(P, Q) = \mathcal{D}/\mathcal{D}H(P, Q).$$

If $P(t) = p\prod_i (t - a_i)$, $Q(t) = q\prod_j (t - b_j)$, $\lambda = p/q$, we shall write

$$\mathcal{H}_\lambda(a_i, b_j) = \mathcal{D}/\mathcal{D}\left(\lambda \prod_i (D - a_i) - z \prod_j (D - b_j)\right).$$

* Corresponding author.
E-mail addresses: golyshev@mccme.ru (V. Golyshev), mellit@gmail.com (A. Mellit).

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Proposition. (i) *$\mathcal{H}_\lambda(a_i, b_j)$ is an irreducible D-module on $\mathbf{G_m}$ if and only if P and Q have no common zeros* (mod $\mathbb{Z}$).

(ii) *Let $\mathcal{H}_\lambda(a_i, b_j)$ be an irreducible hypergeometric D-module. If $\#\{a_i\} \neq \#\{b_j\}$ then $\mathcal{H}_\lambda(a_i, b_j)$ is a differential equation on $\mathbf{G_m}$. If $\#\{a_i\} = \#\{b_j\}$, put $U = \mathbf{G_m} \setminus \{\lambda\}$ and let $j : U \to \mathbf{G_m}$ be the respective open immersion. Then $\mathcal{H}_\lambda(a_i, b_j)$ is a differential equation on U, $\mathcal{H}_\lambda(a_i, b_j) = j_{!*}j^*\mathcal{H}_\lambda(a_i, b_j)$, and the local monodromy of its solutions around λ is a pseudoreflection.*

(iii) *Fix a λ. Let $\mathcal{H}_\lambda(a_i, b_j) = \mathcal{H}(P, Q)$ be an irreducible D-module on $\mathbf{G_m}$. Then the isomorphism class of $\mathcal{H}_\lambda(a_i, b_j)$ depends only on the sets $\{a_i \bmod \mathbb{Z}\}$ and $\{b_j \bmod \mathbb{Z}\}$.*

(iv) *Let $\mathcal{H}_\lambda(a_i, b_j) = \mathcal{H}(P, Q)$ be an irreducible D-module on $\mathbf{G_m}$ of type (n, m). If $n \geq m$ respectively, $m \geq n$, then the eigenvalues of the local monodromy at the regular singularity 0 are $\exp(2\pi i a)_{P(a)=0}$ (resp., the eigenvalues of the local monodromy at the regular singularity ∞ are $\exp(2\pi i b)_{Q(b)=0}$); to each eigenvalue of the local monodromy at 0 (resp. at ∞) corresponds the unique Jordan block.*

(v) *The isomorphism class $\mathcal{H}_\lambda(a_i, b_j)$ determines the type (n, m), the sets $\{a_i \bmod \mathbb{Z}\}$, $\{b_j \bmod \mathbb{Z}\}$ with multiplicities and, in the $n = m$ case, the scalar λ.*

(vi) *Let $\mathcal{F}$, $\mathcal{G}$ be two irreducible local systems on $(\mathbf{G_m} \setminus \{\lambda\})^{an}$ of the same rank $n \geq 1$. Assume that:*

(a) *the local monodromies of both systems at λ are pseudoreflections;*

(b) *the characteristic polynomials of the local systems $\mathcal{F}$ and $\mathcal{G}$ at 0 are equal;*

(c) *the characteristic polynomials of the local systems $\mathcal{F}$ and $\mathcal{G}$ at ∞ are equal.*

Then there is an isomorphism $\mathcal{F} \cong \mathcal{G}$.

We say that a holomorphic function $\Phi(s)$ is of Paley–Wiener type if it is a Fourier transform of a function/distribution H on $\mathbb{R}$ with compact support. Assume that $\Phi(s)$ satisfies a linear homogeneous recurrence R with polynomial coefficients. Then, for any periodic distribution $p(s)$, the product $p(s)\Phi(s)$ satisfies R as well, its inverse Fourier transform being a solution to the DE that is the inverse Fourier transform of R. In particular, let $p = \Delta^t = \sum_{l\in\mathbb{Z}} \delta_{t+l}$. We thus get a system of solutions S_t.

Define now a *balanced gamma product* by

$$\mathbf{\Gamma}(s) = \frac{1}{\prod_{i=1}^{n} \Gamma(s - \alpha_i + 1) \prod_{j=1}^{n} \Gamma(-s + \beta_j + 1)}.$$

Then $\mathbf{\Gamma}(s)$ is of Paley–Wiener type, and the considerations above apply. The corresponding hypergeometric equation is $\mathcal{H}_\lambda(\alpha_i, \beta_i)$ with $\lambda = (-1)^n$. The inverse Fourier transform h of $\mathbf{\Gamma}(s)$ is a solution of the hypergeometric equation on the universal cover of the unit circle. This solution is supported on $[-\frac{n}{2}, \frac{n}{2}]$, which is a union of n segments of length 1, each segment being identified with the unit circle without the singular point. Thus we obtain n solutions of the equation on the unit circle, which form a basis, which we denote by f, and a one-parameter family of solutions S_t.

Looked at from this viewpoint, the non-resonant hypergeometric monodromy (i.e. one with distinct α's and β's mod $\mathbb{Z}$) can be computed easily as follows. Construct a basis of solutions given by the power series in the neighborhood of 0, and similarly in the neighborhood of ∞. In the notation adopted above, the former basis is simply $\{S_{\alpha_k}\}$, and the latter, $\{S_{\beta_{k'}}\}$. It is clear that

$$S_{\alpha_k} = \sum f_m \exp(2\pi i([n/2] - m)\alpha_k) \quad \text{and} \quad S_{\beta_{k'}} = \sum f_m \exp(2\pi i([n/2] - m)\beta_{k'}).$$

In the basis S_α the monodromy around 0 is diagonal with eigenvalues $\exp(2\pi i\alpha_k)$, in the basis S_β the monodromy around ∞ is diagonal with eigenvalues $\exp(2\pi i\beta_{k'})$, and the relation above of each basis to f is a means to glue up the two. The present paper is an elaboration of this concise argument in a possibly resonant case.

Put $\partial = \frac{1}{2\pi i}\frac{\partial}{\partial t}$. We take the liberty of denoting this derivation also as $\frac{\partial}{2\pi i\partial t}$. In a resonant case, one considers derived periodic distributions $p = (-1)^r \partial^r \Delta^t$. A *replication* of the inverse Fourier transform h of the balanced gamma–product is the resulting inverse Fourier transform of $\boldsymbol{\Gamma} \cdot p$. The *gamma structure* $[\boldsymbol{\Gamma}]$ on $\mathcal{H}_\lambda(\alpha, \beta)$ associated with $\boldsymbol{\Gamma}$ is defined to be the set of all replications of h. Theorem 5.8 states that the hypergeometric monodromy expressed in terms of the basis of local solutions at 0 (resp. ∞) that are in the gamma structure is given by products of generalized diagonal matrices and Vandermonde matrices (and their inverses), whose entries depend algebraically on the hypergeometric exponents, cf. [2].

1. The Paley–Wiener property of gamma products

Proposition 1.1. *There exists $C > 0$ such that for all $s \in \mathbb{C}$*

$$\left|\frac{1}{\Gamma(s)}\right| < C(1 + |s|)^{\frac{1}{2} - \operatorname{Re} s}\, e^{\arg s\, \operatorname{Im} s + \operatorname{Re} s},$$

where $\arg s$ *is chosen to be in* $[-\pi, \pi]$.