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Poisson and Hamiltonian structures on complex analytic foliated manifolds

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ABSTRACT

Poisson and Hamiltonian structures are introduced in the category of complex analytic foliated manifolds endowed with a hermitian metric by analogy with the case of real foliated manifolds studied by Vaisman. A particular case of Hamiltonian structure, called tame, is proved to be induced by a Poisson bracket on the underlying manifold.

1. Introduction

Short time after their introduction in the real case [1,2] the Poisson structures on manifolds were studied in the complex (holomorphic) context in [3–10].

In [11–13], I. Vaisman suggests that it is interesting to study Hamiltonian, Poisson and related structures on foliated manifolds since these may be relevant to the study of physical systems depending on gauge parameters, which are the co-ordinates along the leaves of a foliation. The aim of our present paper is to extend the study of such structures from the mathematical point of view on complex analytic foliated manifolds endowed with a hermitian metric. This metric allows us to consider an orthogonal complement of the holomorphic tangent bundle of given foliation and therefore yields a graded calculus with vector fields, differential forms and bivectors.

The paper is structured as follows. First, following [14–16] we briefly recall the Vaisman complex of differential forms of mixed type and some preliminary notions about calculus on complex analytic foliated manifolds. Next, we define transversally Poisson and Hamiltonian structures related to our considerations and such a later structure defines a Poisson structure

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on the algebra of complex analytic foliated functions. It is shown that in the so-called tame case, the Hamiltonian structure is induced by an usual Poisson structure of the manifold.

2. Preliminaries on complex analytic foliations

Let us begin our study with a short review of complex analytic foliated manifolds and set up the basic notions and terminology. For more details see [14–16].

Definition 2.1. A *complex analytic foliated* structure $\mathcal{F}$, briefly c.a.f., of complex codimension n on a complex $(n+m)$-dimensional manifold M is given by an atlas $\{\mathcal{U}, (z_\alpha^a, z_\alpha^u)\}$, $a, b, \ldots = 1, \ldots, n$; $u, v, \ldots = n+1, \ldots, n+m$, such that for every $U_\alpha, U_\beta \in \mathcal{U}$ with $U_\alpha \cap U_\beta \neq \phi$, one has, besides analyticity: $\partial z_\beta^a / \partial z_\alpha^u = 0$.

Then the maximal connected submanifolds which can be represented locally by $z_\alpha^a = \text{const.}$ are the *leaves* of $\mathcal{F}$ and the images $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ of the submersions $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ defined by $\varphi_\alpha(z_\alpha^a, z_\alpha^u) = (z_\alpha^a)$ are called the *local transverse manifolds*.

Let $T'M$ be the holomorphic tangent bundle of M and $T''M = \overline{T'M}$ the antiholomorphic tangent bundle respectively. The tangent vectors of the leaves define the *structural subbundle* $V'\mathcal{F} = T'\mathcal{F}$ of $T'M$ with local bases $\{Z_u = \partial/\partial z_\alpha^u\}$ and the transition functions $(\partial z_\beta^u / \partial z_\alpha^v)$, and $Q'\mathcal{F} = T'M/V'\mathcal{F}$ is the *transversal bundle* with the local bases defined by the equivalence classes $[\partial/\partial z_\alpha^a]$ and the transition functions $(\partial z_\beta^a / \partial z_\alpha^b)$.

Generally, we shall say that the geometrical objects depending only on the leaves are *foliated* and, particularly, c.a.f. For instance, $f : M \to \mathbb{C}$ is foliated if $\partial f/\partial z_\alpha^u = \partial f / \partial \overline{z}_\alpha^u = 0$ and it is c.a.f. if, moreover, $\partial f / \partial \overline{z}_\alpha^a = 0$. A differential form is c.a.f. if it does not contain $d\overline{z}_\alpha^a, dz_\alpha^u, d\overline{z}_\alpha^u$ and has local c.a.f. coefficients. A vector bundle on M is c.a.f. if it has c.a.f. transition functions; for instance, the transversal bundle is such.

In the following, let us suppose that M is hermitian with metric h. Then the orthogonal bundle $H'\mathcal{F} = (T'\mathcal{F})^\perp$ of $T'\mathcal{F}$, i.e. $T'M = H'\mathcal{F} \oplus V'\mathcal{F}$, which is differentially isomorphic to $Q'\mathcal{F}$, has local bases of the form:

$$Z_a = \frac{\partial}{\partial z^a} - t_a^u(z^a, z^u) \frac{\partial}{\partial z^u} \tag{2.1}$$

(the index α of the coordinate neighborhood will be omitted) and we shall use in the sequel the bases $\{Z_a, Z_u\}$ to express different vector fields of $\mathcal{X}(M)$. Also, we shall simply denote $H' := H'\mathcal{F}$ and $V' := V'\mathcal{F}$, respectively and by conjugation, we have $V'' = \overline{V'}$ and $H'' = \overline{H'}$ where $V'' = \text{span}\{\overline{Z_u}\}$ and $H'' = \text{span}\{\overline{Z_a}\}$ respectively. Then we have a decomposition of the complexified tangent bundle of $(M, \mathcal{F})$, namely $T_{\mathbb{C}}M = H \oplus V$, where $H = H' \oplus H''$ and $V = V' \oplus V''$. The corresponding dual cobases are given by:

$$\{dz^a\},\ \{\theta^u = dz^u + t_a^u dz^a\},\ \{d\overline{z}^a\},\ \{\overline{\theta}^u = d\overline{z}^u + \overline{t_a^u} d\overline{z}^a\}, \tag{2.2}$$

which span the dual bundles $H'^* = ann\{H'' \oplus V\}$, $H''^* = ann\{H' \oplus V\}$, $V'^* = ann\{H \oplus V''\}$ and $V''^* = ann\{H \oplus V'\}$ respectively. Here, by *ann* we denote the annihilator of a vector bundle. Now, let $\Omega(M)$ and $\mathcal{V}(M)$ be the exterior algebras of differential forms and multivector fields on M respectively.

The cobases from (2.2) allow us to speak of the type (p_1, p_2, q_1, q_2) of a differential form by counting in its expression the number of $dz^a, d\overline{z}^a, \theta^u$ and $\overline{\theta}^u$, respectively. Thus, we denote by $\Omega^{p_1,p_2,q_1,q_2}(M) = \bigwedge^{p_1,p_2} H \wedge \bigwedge^{q_1,q_2} V$ the set of all (p_1, p_2, q_1, q_2)-differential forms, locally given by:

$$\varphi = \frac{1}{p_1!p_2!q_1!q_2!} \sum \varphi_{A_{p_1}\overline{B}_{p_2}U_{q_1}\overline{V}_{q_2}} dz^{A_{p_1}} \wedge d\overline{z}^{B_{p_2}} \wedge \theta^{U_{q_1}} \wedge \overline{\theta}^{V_{q_2}} \tag{2.3}$$

where $A_{p_1} = (a_1 \ldots a_{p_1})$, $B_{p_2} = (b_1 \ldots b_{p_2})$, $U_{q_1} = (u_1 \ldots u_{q_1})$, $V_{q_2} = (v_1 \ldots v_{q_2})$, $dz^{A_{p_1}} = dz^{a_1} \wedge \cdots \wedge dz^{a_{p_1}}$, $d\overline{z}^{B_{p_2}} = d\overline{z}^{b_1} \wedge \cdots \wedge d\overline{z}^{b_{p_2}}$, $\theta^{U_{q_1}} = \theta^{u_1} \wedge \cdots \wedge \theta^{u_{q_1}}$ and $\overline{\theta}^{V_{q_2}} = \overline{\theta}^{v_1} \wedge \cdots \wedge \overline{\theta}^{v_{q_2}}$ respectively.

These forms can be considered as *complex type* (p_1+q_1, p_2+q_2), as *foliated type* (p_1+p_2, q_1+q_2) and as *mixed type* $(p_1, p_2+q_1+q_2)$, respectively.

Throughout this paper we consider forms of mixed type and we denote by $\Omega^{p,q}_{\text{mix}}(M, \mathcal{F})$ the space of all differential forms of mixed type (p, q) on $(M, \mathcal{F})$. According to above discussion we have:

$$\Omega^{p,q}_{\text{mix}}(M, \mathcal{F}) = \bigoplus_{r,h} \Omega^{p,r,h,q-r-h}(M, \mathcal{F}) = \bigoplus_{k=0}^{q} \bigwedge^{p,k} H \wedge \bigwedge^{q-k}(V, \mathbb{C}).$$

The metric can be locally expressed by:

$$h = h_{a\overline{b}} dz^a \otimes d\overline{z}^b + h_{u\overline{v}} \theta^u \otimes \overline{\theta}^v \tag{2.4}$$