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Lorentzian compact manifolds: Isometries and geodesics

Viviana del Barco [a,*], Gabriela P. Ovando [b], Francisco Vittone [a]

[a] Facultad de Ciencias Exactas Ingeniería y Agrimensura, Universidad Nacional de Rosario, Av. Pellegrini 250, (2000) Rosario, Argentina

[b] CONICET—Facultad de Ciencias Exactas Ingeniería y Agrimensura, Universidad Nacional de Rosario, Av. Pellegrini 250, (2000) Rosario, Argentina

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ABSTRACT

In this work we investigate families of compact Lorentzian manifolds in dimension four. We show that every lightlike geodesic on such spaces is periodic, while there are closed and non-closed spacelike and timelike geodesics. Also their isometry groups are computed. We also show that there is a non trivial action by isometries of $\mathrm{H}_3(\mathbb{R})$ on the nilmanifold $S^1 \times (\Gamma_k \backslash \mathrm{H}_3(\mathbb{R}))$ for Γ_k, a lattice of $\mathrm{H}_3(\mathbb{R})$.

1. Introduction

Due to their relations with general relativity Lorentzian manifolds, that is manifolds endowed with metric tensors of index 1, play a special role in pseudo-Riemannian geometry. Timelike and null geodesics represent, respectively, free falling particles and light rays. Isometric actions and the existence problem of closed geodesics are two of the most popular topics of research in the last time. In this work by a closed geodesic we mean a periodic geodesic.

The known results developed in the field have made use of several techniques including variational and topological methods, Lie theory, etc. (See for instance [1–5] and references therein.) After the classification of simply connected Lie groups acting locally faithfully by isometries on a compact Lorentz manifold [6,7] some other questions concerning the geometric implications of such actions arise in a natural way, specially in the noncompact case (see [8]). In [9] Melnick investigated the isometric actions of Heisenberg groups on compact Lorentzian manifolds, showing a codimension one action of the Heisenberg Lie group $\mathrm{H}_3(\mathbb{R})$ on the Lorentzian compact solvmanifold $M = \Gamma\backslash G$, where $G = \mathbb{R} \ltimes \mathrm{H}_3(\mathbb{R})$ is a solvable Lie group, called the oscillator group.

The main purpose of this work is to analyze these topics more deeply in a family of examples. We study the geometry of families of compact Lorentzian manifolds in dimension four: $M_{k,i} = G/\Lambda_{k,i}$, which are stationary, that is, they admit an everywhere timelike Killing vector field. This implies the existence of closed timelike geodesics (see [10]).

* Corresponding author. Tel.: +54 3414802649.
E-mail addresses: delbarc@fceia.unr.edu.ar (V. del Barco), gabriela@fceia.unr.edu.ar (G.P. Ovando), vittone@fceia.unr.edu.ar (F. Vittone).

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In this work we obtain:

- Every lightlike geodesic on any compact space $M_{k,i}$ is periodic, while there are periodic and injective timelike and spacelike geodesics.
- The isometry groups of these compact spaces have a countable amount of connected components (see [11]).

As already mentioned the existence question of closed geodesics on a compact Lorentzian manifold is a classical topic in Lorentzian geometry. In this context the results above relative to null geodesics are surprising in a quite different situation of those in [12] and therefore they should induce new research in the topic.

We start with an isometric codimension one action by isometries of the Heisenberg Lie group $\mathrm{H}_3(\mathbb{R})$ on compact nilmanifolds $\Lambda_k\backslash N$ where $N = \mathbb{R} \times \mathrm{H}_3(\mathbb{R})$. The starting point is the existence of an isometry between the Lorentzian Lie group G which is solvable and the Lie group N which is 2-step nilpotent [13]. This reveals that the existence of actions by isometries coming from non-isomorphic groups does not distinguish the isometry class of the Lorentzian manifold. However while the Lorentzian metric on G is bi-invariant, that one on N is only left-invariant. Furthermore there is a family of groups Λ_k which are cocompact lattices of G and also of N so that every quotient $\Lambda_k\backslash N$ is diffeomorphic to $\Lambda_k\backslash G$ and the metrics induced to the quotients give rise to an isometry between the compact spaces $(\Lambda_k\backslash N, g_N)$ and $(\Lambda_k\backslash G, g_G)$. It is clear that as an ideal of G, the Heisenberg Lie group $\mathbb{H}_3$ acts isometrically on $\Lambda_k\backslash G$ by translations on the right. Therefore the Heisenberg Lie group also acts on $\Lambda_k\backslash N$ by isometries. The Lie group N is already known in the literature: it is related to the known *Kodaira–Thurston* manifold. One of the advantages of the nilmanifold model arises from Nomizu's Theorem: the de Rham cohomology can be read off from the cohomology of the Lie algebra of N.

The solvable group G admits more cocompact lattices $\Lambda_{k,i}$ which are not isomorphic to the family above. We explicitly write the full isometry group of G which is proved to be non-compact. And making use of results which relate the isometries on the quotients with those on G we compute $\mathrm{Iso}(M_{k,i})$ the group of isometries of the compact solvmanifolds $M_{k,i} = \Lambda_{k,i}\backslash G$.

We complete the work with the study of the periodic geodesics on the compact Lorentzian solvmanifolds. It should be noticed that all the Lorentzian manifolds here are naturally reductive spaces. We notice that together with the motivations coming from Lorentzian geometry an active research is given for g.o. spaces (see for instance [14–17]). The compact Lorentzian spaces $M_{k,i}$ constitute the first examples (known to us) of compact spaces in dimension four where every lightlike geodesic is periodic.

2. Lorentzian nilmanifolds and actions

Let $\mathrm{H}_3(\mathbb{R})$ denote the Heisenberg Lie group of dimension three, which modeled over $\mathbb{R}^3$ has a multiplication map given by

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right).$$

Let N denote the nilpotent Lie group $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$, which turns into a pseudo-Riemannian manifold modeled on $\mathbb{R}^4$ with the following Lorentzian metric

$$g = dt\left(dz + \frac{1}{2}ydx - \frac{1}{2}xdy\right) + dx^2 + dy^2 \tag{1}$$

where (t, x, y, z) are usual coordinates for $\mathbb{R}^4$. Denote $v = (x, y)$ and for each $(t_1, v_1, z_1) \in \mathbb{R}^4$ consider the following differentiable function on $\mathbb{R}^4$:

$$L^N_{(t_1,v_1,z_1)}(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^\tau J v_2\right) \tag{2}$$

where J is the linear map on $\mathbb{R}^2$ given by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Clearly L^N is the translation on the left on N by the element (t_1, v_1, z_1) and it is not hard to see that the metric g is invariant under the left-translations $L^N_{(t_1,v_1,z_1)}$. A basis of left-invariant vector fields at $p = (t, x, y, z)$ is

$$e_0(p) = \partial_t|_p$$
$$e_1(p) = \partial_x|_p - \frac{1}{2}\, y\, \partial_z\Big|_p$$
$$e_2(p) = \partial_y|_p + \frac{1}{2}\, x\, \partial_z\Big|_p$$
$$e_3(p) = \partial_z|_p$$

and the invariant Lorentzian metric g satisfies

$$g(e_0, e_3) = g(e_1, e_1) = g(e_2, e_2) = 1.$$