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Constraints and symmetry in mechanics of affine motion

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ABSTRACT

The aim of this paper is to perform a deeper geometric analysis of problems appearing in dynamics of affinely rigid bodies. First of all we present a geometric interpretation of the polar and the two-polar decomposition of affine motion. Later on some additional constraints imposed on the affine motion are reviewed, both holonomic and non-holonomic. In particular, we concentrate on certain natural non-holonomic models of the rotation-less motion. We discuss both the usual d'Alembert model and the vakonomic dynamics. The resulting equations are quite different. It is not yet clear which model is practically better. In any case they both are different from the holonomic constraints defining the rotation-less motion as a time-dependent family of symmetric matrices of placements. The latter model seems to be non-geometric and non-physical. Nevertheless, there are certain relationships between our non-holonomic models and the polar decomposition.

0. Introduction

There is plenty of misunderstandings concerning constraints and symmetries in mechanics of continuous media. Many of them are additionally obscured by problems on the level of functional analysis characteristic for the field theory and any similar scheme with infinite number of degrees of freedom. Because of this it is convenient to try to discuss the problems on the level of discretized continuum with a finite number of degrees of freedom. The most natural model is an affinely-rigid or pseudo-rigid body. By this we mean the body, the deformative behaviour of which is restricted to homogeneous deformations [1]. It is an important model because locally, in an infinitesimal neighbourhood of any point, every smooth deformation is homogeneous. In any case, it is sufficient to the analysis of the spatial and material invariance of any mechanical problem, at least for simple materials. Besides, the affine model of degrees of freedom enables one to achieve a deeper understanding of the constrained dynamics, especially when the constraints have some readable group-theoretical background. In particular, besides more traditional holonomic constraints, we discuss in some detail the case of rotation-less motion. One can think, for instance, about applications in the theory of motion of small deformable inclusions in a viscous fluid. In some popular rough approaches one often claims that the configuration is rotation-less when the placement matrix is symmetric. We show that this concept is incorrect because of two reasons: (1) symmetric matrices do not form a group, therefore the relationship would not be transitive, and (2) symmetry of the placement matrix is a non-geometric concept, because the tensor indices refer to different spaces, the physical and material ones. However, as we show, one can reasonably tell about the rotation-less motion. By this we mean such a motion that the Eringen gyration is symmetric with respect to

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the metric tensor (in the metrically-rigid motion it is skew-symmetric, just the angular velocity). But such constraints are essentially non-holonomic. It is well known today that there exist two kinds of non-holonomic dynamics, one based on the d'Alembert variational principle and one based on the Lusternik variational principle, so-called vakonomy [2–5]. We review both of them. They are non-equivalent, but it is still not clear which of them may be used in realistic mechanical problems. It is important that there are two types of non-holonomic rotation-less constraints: spatially non-rotational and materially non-rotational. They are non-equivalent, unlike holonomic gyroscopic constraints which exist only in one form. We concentrate here on spatially rotation-less constraints, although materially rotation-less ones are also briefly described.

1. Affine constraints, geometry of the polar and two-polar decompositions

Let us begin with a short review of our earlier results concerning the mechanics of affinely-rigid body [6–8]. To be honest, some of them are also partially contained in Eringen's theory of micromorphic media, i.e., continua of infinitesimal affine bodies [9]. Later on, we developed the theory in various aspects [10–31] and some of our results were confirmed and developed by many people [32–39]. Let us also mention the papers like [40–45]. Nevertheless, in spite of numerous applications the topic does not belong to commonly known standards, and because of this a brief repetition seems to be necessary.

Let us consider a system of material points moving in n-dimensional physical space M; we assume M to be an affine space with the linear space of translations V, endowed also with the symmetric and positively-definite metric tensor $g \in V^* \otimes V^*$. If necessary, the translation vector from $x \in M$ to $y \in M$ will be denoted by $\overrightarrow{xy}$. The material space, i.e., the set of material points will be also an affine space N of the same dimension n, with the linear space of translations U. The material metric tensor will be denoted by $\eta \in U^* \otimes U^*$, and translation vectors by $\overrightarrow{ab}$ for $a, b \in U$. As usual, we say that a mapping $\phi : N \to M$ is affine if it preserves all affine relationships, i.e., there exists a linear mapping $L[\phi] : U \to V$, denoted also as $D\phi \in L(U, V)$ such that

$$\overrightarrow{\phi(a)\,\phi(b)} = L[\phi]\,\overrightarrow{ab} \tag{1.1}$$

for any pair of material points, $a, b \in N$. If y^i, a^K are affine coordinates respectively in M and N, this means obviously that ϕ is analytically given by first-order polynomials:

$$y^i = x^i + \varphi^i{}_K a^K. \tag{1.2}$$

Obviously, this definition is valid for any, not necessarily equal dimensions of N, M. The set of all affine mappings of N onto M will be denoted by $\mathrm{Aff}(N, M)$, and the set of all one-to-one affine mappings of N onto M is denoted by $\mathrm{AffI}(N, M)$ (affine isomorphisms). Obviously, $\mathrm{AffI}(N, M)$ is non-empty only if $\dim N = \dim M$, and for any $\phi \in \mathrm{AffI}(N, M)$, $\varphi = L[\phi] \in LI(U, V)$, i.e., it is a linear isomorphism of U onto V. The groups of affine and linear isomorphisms of M and V will be denoted by $GAff(M)$, $GL(V)$. They are open subsets of $\mathrm{Aff}(M)$, $L(V)$, i.e., of the sets of all affine and linear mappings of M and V into themselves.

Every choice of affine coordinates a^K, y^i in N, M pre-assumes two things: a choice of the origins $\mathfrak{O} \in N$, $\mathfrak{o} \in M$ of coordinates in N, M and a choice of bases $(\ldots, E_A, \ldots)$, $(\ldots, e_i, \ldots)$ in U, V, or equivalently, a choice of dual bases $\left(\ldots, E^A, \ldots\right)$, $\left(\ldots, e^i, \ldots\right)$ in U^*, V^*. Then we have

$$a^K(P) = \left\langle E^K, \overrightarrow{\mathfrak{O}P}\right\rangle, \qquad y^i(p) = \left\langle e^i, \overrightarrow{\mathfrak{o}p}\right\rangle \tag{1.3}$$

for any points $P \in N$, $p \in M$. When the constant co-moving mass distribution in N is fixed and described by positive measure μ on N, then it is natural to choose $\mathfrak{O} \in N$ as the centre of mass,

$$\int \overrightarrow{\mathfrak{O}P}\, d\mu(P) = 0. \tag{1.4}$$

The point $\mathfrak{O}$ is uniquely defined when $m = \mu(N)$ is finite, what is physically always assumed. With such a choice of $\mathfrak{O}$, the quantities x^i in (1.2) are the current coordinates of the centre of mass in M, $\mathfrak{o}_\phi = \phi(\mathfrak{O})$. Let us stress that for any, not necessarily affine, configuration $\mathfrak{o}_\phi$ is defined by the condition

$$\int \overrightarrow{\mathfrak{o}_\phi p}\, d\mu_\phi(p) = 0, \tag{1.5}$$

where μ_ϕ denotes the ϕ-transport of the measure μ from N to M. The equality $\mathfrak{o}_\phi = \phi(\mathfrak{O})$ holds only for affine configurations.

When the choice of $\mathfrak{O}$ is fixed as above, then the configuration space of affinely-rigid body, i.e., the manifold of affine isomorphisms of N onto M, $\mathrm{AffI}(N, M)$ becomes canonically identified with the Cartesian product $M \times LI(U, V)$:

$$\phi \equiv (\phi(\mathfrak{O}), L[\phi]) = \left(\ldots, x^i, \ldots; \ldots, \varphi^i{}_K, \ldots\right). \tag{1.6}$$

This is the splitting of degrees of freedom into translational and internal ones.