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Sasaki–Einstein and paraSasaki–Einstein metrics from (κ, μ)-structures

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ABSTRACT

We prove that every contact metric (κ, μ)-space admits a canonical η-Einstein Sasakian or η-Einstein paraSasakian metric. An explicit expression for the curvature tensor fields of those metrics is given and we find the values of κ and μ for which such metrics are Sasaki–Einstein and paraSasaki–Einstein. Conversely, we prove that, under some natural assumptions, a K-contact or K-paracontact manifold foliated by two mutually orthogonal, totally geodesic Legendre foliations admits a contact metric (κ, μ)-structure. Furthermore, we apply the above results to the geometry of tangent sphere bundles and we discuss some geometric properties of (κ, μ)-spaces related to the existence of Einstein–Weyl and Lorentzian–Sasaki–Einstein structures.

1. Introduction

It is well known that the tangent sphere bundle T_1N of a flat Riemannian manifold N carries a contact Riemannian structure such that $R(X, Y)\xi = 0$ for any vector fields X, Y on T_1N, where the Reeb vector field ξ is given by twice the geodesic flow. The class of contact metric manifolds satisfying the above condition, which were at first studied by Blair in [1], is not preserved by $\mathcal{D}$-homothetic transformations. In fact, if one deforms $\mathcal{D}$-homothetically the structure, one falls in the larger class of "contact metric (κ, μ)-spaces", i.e. contact metric manifolds $(M, \varphi, \xi, \eta, g)$ satisfying

$$R(X, Y)\xi = \kappa\left(\eta(Y)X - \eta(X)Y\right) + \mu\left(\eta(Y)hX - \eta(X)hY\right), \tag{1.1}$$

for some constants κ and μ, where $2h$ denotes the Lie derivative of the structure tensor φ in the direction of the Reeb vector field (see Section 2 for more details). This new class of Riemannian manifolds was introduced in [2] as a natural

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generalization of both the contact metric manifolds satisfying $R(X, Y)\xi = 0$ and the Sasakian condition $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$. Despite the technical appearance of the definition, nowadays contact (κ, μ)-spaces are considered an important topic in contact Riemannian geometry because there are good reasons for studying them. The first is that, while the values κ and μ vary, one proves that the condition (1.1) remains unchanged under $\mathcal{D}$-homothetic deformations. Next, in the non-Sasakian case (that is for $\kappa \neq 1$), the condition (1.1) determines the curvature tensor field completely. Furthermore, (κ, μ)-spaces provide non-trivial examples of some remarkable classes of contact Riemannian manifolds, like CR-integrable contact metric manifolds [3], H-contact manifolds [4] and harmonic contact metric manifolds [5]. Finally, there are non-trivial examples of such Riemannian manifolds, the most important being the tangent sphere bundle of any Riemannian manifold of constant sectional curvature with its standard contact metric structure.

In this paper we study the relations between the theory of (κ, μ)-spaces and two other important topics of contact geometry: Sasakian and paraSasakian manifolds. In fact, given a non-Sasakian (κ, μ)-space $(M, \varphi, \xi, \eta, g)$, we describe a method for constructing a Sasakian or paraSasakian metric on M compatible with the same contact form η. The type of metric (Sasakian or paraSasakian) depends on the value of a well-known invariant introduced by Boeckx in [6] for classifying (κ, μ)-spaces, defined as

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}}. \tag{1.2}$$

More precisely, we are able to define a Sasakian or paraSasakian metric if $|I_M| > 1$ or $|I_M| < 1$, respectively. Moreover, by using the aforementioned property that the (κ, μ)-nullity condition (1.1) determines the curvature completely, we find an explicit expression for the curvature tensor field of the above Sasakian and paraSasakian metrics. We obtain from it our main result that such metrics are always η-Einstein and that for some values of κ and μ they are Sasaki–Einstein and paraSasaki–Einstein, though the starting (κ, μ)-structure can never be Einstein in dimension greater than 3 [7, p. 131]. Furthermore, we prove that in dimension greater than or equal to 5, every (κ, μ)-space such that $I_M > 1$ also carries an Einstein–Weyl structure.

We then discuss some consequences of such results on the geometry of tangent sphere bundles T_1N, which will accept η-Einstein Sasakian and paraSasakian metrics depending on the sign of c, the constant sectional curvature of the space form N. Moreover, these structures will be Sasaki–Einstein and paraSasaki–Einstein for certain values of c (which will depend only on n). Thus we extend the result of Tanno that $T_1S^3 \simeq S^2 \times S^3$ carries a Sasaki–Einstein metric and give (to the knowledge of the authors) the first non-trivial examples of η-Einstein (eventually Einstein) paraSasakian manifolds. In fact, while there has been increasing interest in paraSasakian geometry in the last years (see [8–10]), so far the only known examples of (η-)Einstein paracontact manifolds seem to be the hyperboloid $\mathbb{H}^{2n+1}_{n+1}(1)$ of constant curvature -1 [9] and $\mathbb{R}^3_1$ with the flat metric [11], together with the Boothby–Wang fibrations with base a paraKähler–Einstein manifold.

Finally, in the last part of the paper we will give a geometric interpretation to the above canonical Sasakian and paraSasakian metrics. It is well known that any non-Sasakian (κ, μ)-space is foliated by two Legendre foliations, defined by the eigendistributions of the operator h, and that such a foliated structure plays an important role in the theory of (κ, μ)-spaces (cf. [12,13]). We show that the geometry of these Legendre foliations, encoded by some invariants like the Pang invariant [14] and the Libermann map [15], is fully described by the above Sasakian and paraSasakian metrics. In this way we are able to find a sufficient condition for a K-contact (respectively, K-paracontact) manifold M, foliated by two mutually orthogonal, totally geodesic Legendre foliations, to admit a contact metric (κ, μ)-structure, compatible with the same underlying contact form, such that $|I_M| > 1$ (respectively, $|I_M| < 1$).

2. Preliminaries

2.1. Contact metric (κ, μ)-spaces

An *almost contact structure* on a $(2n + 1)$-dimensional smooth manifold M is a triplet (φ, ξ, η), where φ is a tensor field of type $(1, 1)$, η a 1-form and ξ a vector field on M satisfying the following conditions:

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

where I is the identity mapping. From (2.1) it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the $(1, 1)$-tensor field φ has constant rank $2n$ [7]. Given an almost contact manifold (M, φ, ξ, η) one can define an almost complex structure J on the product $M \times \mathbb{R}$ by setting $J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$ for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$. Then the almost contact manifold is said to be *normal* if the almost complex structure J is integrable. The condition for normality is given by the vanishing of the tensor field $N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi$. Any almost contact manifold (M, φ, ξ, η) admits a *compatible metric*, i.e. a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$. The manifold M is said to be an *almost contact metric manifold* with structure (φ, ξ, η, g). From (2.2) it follows immediately that $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)$. Then one defines the 2-form Φ on M by $\Phi(X, Y) = g(X, \varphi Y)$,