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Pseudo-Riemannian spectral triples and the harmonic oscillator

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ABSTRACT

We define pseudo-Riemannian spectral triples, an analytic context broad enough to encompass a spectral description of a wide class of pseudo-Riemannian manifolds, as well as their noncommutative generalisations. Our main theorem shows that to each pseudo-Riemannian spectral triple we can associate a genuine spectral triple, and so a K-homology class. With some additional assumptions we can then apply the local index theorem. We give a range of examples and some applications. The example of the harmonic oscillator in particular shows that our main theorem applies to much more than just classical pseudo-Riemannian manifolds.

1. Introduction

Spectral triples provide a way to extend Riemannian geometry to noncommutative spaces, retaining the connection to the underlying topology via K-homology. In this paper we provide a definition of pseudo-Riemannian spectral triple, enabling a noncommutative analogue of pseudo-Riemannian geometry, and show that we can still maintain contact with the underlying topology. We do this by 'Wick rotating' to a spectral triple analogue of our pseudo-Riemannian spectral triple. Below we discuss the class of pseudo-Riemannian manifolds which give examples of our construction: a key point is that we work with Hilbert spaces not Krein spaces, and so the construction relies on a global splitting of the tangent bundle into timelike and spacelike sub-bundles.

Our main theorem, Theorem 5.1, states that one can associate a spectral triple to a pseudo-Riemannian spectral triple via Wick rotation. Under additional assumptions, the process of Wick rotating is shown to preserve spectral dimension, smoothness and integrability, as we define them. Thus one obtains a K-homology class and the tools to compute index pairings using the local index formula. Since the most important Lorentzian manifolds are noncompact, we have taken care to ensure that our definitions are consistent with the nonunital version of the local index formula, as proved in [1].

Section 2 recalls what we need from the theory of nonunital spectral triples, while Section 3 recalls some pseudo-Riemannian geometry, in order to set notation and provide motivation. Here we also show how certain pseudo-Riemannian spin manifolds provide examples for our theory.

In Section 4 we discuss some technicalities about unbounded operators before presenting our definition of pseudo-Riemannian spectral triples. We also provide definitions of smoothness and summability, and a range of examples. An unexpected example is provided by the harmonic oscillator.

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Section 5 begins with our main theorem, which shows that we can obtain a spectral triple from a pseudo-Riemannian spectral triple. We also give a sufficient condition on a smoothly summable pseudo-Riemannian spectral triple ensuring that the resulting spectral triple is smoothly summable, so that we can employ the local index formula. This sufficient condition is enough for all our examples, except the harmonic oscillator. The remainder of the section looks at the examples, in particular the oscillator, as well as one simple non-existence result for certain kinds of harmonic one forms on compact manifolds.

The classical examples we have presented all arise by taking a pseudo-Riemannian spin manifold and generating a Riemannian metric: for this to work, we require a global splitting of the tangent bundle into timelike and spacelike subbundles, and that the resulting Riemannian metric be complete and of bounded geometry. This can always be achieved in the globally hyperbolic case when $M_n = \mathbb{R} \times M_{n-1}$ provided that the induced metric on M_{n-1} is complete and of bounded geometry.

It would be desirable to have a method of producing a spectral triple, and so K-homology class, associated to a more general pseudo-Riemannian metric, or at least Lorentzian metric. The reason we cannot is that, in the Lorentzian case, the weakest physically reasonable causality condition is stable causality, and the Lorentzian metrics of such manifolds need not have complete Riemannian manifolds associated to them by our Wick rotation procedure.

Hence to deal with general Lorentzian manifolds, one would have to be able to deal with Riemannian manifolds with boundary, and even more general objects. This requires careful consideration of appropriate boundary conditions, and we refer to [2] for a comprehensive discussion of boundary conditions in K-homology and [3] for a definition of 'spectral triple with boundary'. We will return to this question in a future work.

As a final comment, we remark that the purpose of this paper is to develop a framework for accessing the topological data associated to a pseudo-Riemannian manifold, commutative or noncommutative. The details of the geometry and/or physics of such a manifold should be accessed using the pseudo-Riemannian structure and not the associated Riemannian one.

2. Nonunital spectral triples

In this section we will summarise the definitions and results concerning nonunital spectral triples. Much of this material is from [1] where a more detailed account can be found.

Definition 2.1. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by

(1) A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a $*$-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$.
(2) A self-adjoint (unbounded, densely defined) operator $\mathcal{D} : \operatorname{dom} \mathcal{D} \to \mathcal{H}$ such that $[\mathcal{D}, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ and $\pi(a)(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$.

The triple is said to be even if there is an operator $\Gamma = \Gamma^*$ such that $\Gamma^2 = 1$, $[\Gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$ (i.e. Γ is a $\mathbb{Z}_2$-grading such that $\mathcal{D}$ is odd and $\pi(\mathcal{A})$ is even). Otherwise the triple is called odd.

Remark 2.2. We will systematically omit the representation π in future.

Our aim is to define and study generalisations of spectral triples which model pseudo-Riemannian manifolds. The interesting situation in this setting is when the underlying space is noncompact. In order to discuss summability in this context, we recall a few definitions and results from [1], where a general definition of summability in the nonunital/noncompact context was developed.

Definition 2.3. Let $\mathcal{D}$ be a densely defined self-adjoint operator on the Hilbert space $\mathcal{H}$. Then for each $p \geq 1$ and $s > p$ we define a weight φ_s on $\mathcal{B}(\mathcal{H})$ by

$$\varphi_s(T) := \operatorname{Trace}((1 + \mathcal{D}^2)^{-s/4} T (1 + \mathcal{D}^2)^{-s/4}), \quad 0 \leq T \in \mathcal{B}(\mathcal{H}),$$

and the subspace $\mathcal{B}_2(\mathcal{D}, p)$ of $\mathcal{B}(\mathcal{H})$ by

$$\mathcal{B}_2(\mathcal{D}, p) := \bigcap_{s>p} \left(\operatorname{dom}(\varphi_s)^{1/2} \bigcap (\operatorname{dom}(\varphi_s)^{1/2})^* \right).$$

The norms

$$\mathcal{B}_2(\mathcal{D}, p) \ni T \mapsto \mathcal{Q}_n(T) := \left(\|T\|^2 + \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(|T^*|^2) \right)^{1/2}, \quad n = 1, 2, 3 \ldots, \tag{1}$$

take finite values on $\mathcal{B}_2(\mathcal{D}, p)$ and provide a topology on $\mathcal{B}_2(\mathcal{D}, p)$ stronger than the norm topology. We will always suppose that $\mathcal{B}_2(\mathcal{D}, p)$ has the topology defined by these norms.

The space $\mathcal{B}_2(\mathcal{D}, p)$ is in fact a Fréchet algebra, [1, Proposition 2.6], and plays the role of bounded square integrable operators. Next we introduce the bounded integrable operators.

On $\mathcal{B}_2(\mathcal{D}, p)^2$, the span of products TS, with $T, S \in \mathcal{B}_2(\mathcal{D}, p)$, define norms

$$\mathcal{P}_n(T) := \inf\left\{ \sum_{i=1}^{k} \mathcal{Q}_n(T_{1,i})\, \mathcal{Q}_n(T_{2,i}) : T = \sum_{i=1}^{k} T_{1,i} T_{2,i},\ T_{1,i},\ T_{2,i} \in \mathcal{B}_2(\mathcal{D}, p) \right\}. \tag{2}$$