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Nonholonomic LL systems on central extensions and the hydrodynamic Chaplygin sleigh with circulation

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ABSTRACT

We consider nonholonomic systems whose configuration space is the central extension of a Lie group and have left invariant kinetic energy and constraints. We study the structure of the associated Euler–Poincaré–Suslov equations and show that there is a one-to-one correspondence between invariant measures on the original group and on the extended group. Our results are applied to the hydrodynamic Chaplygin sleigh, that is, a planar rigid body that moves in a potential flow subject to a nonholonomic constraint modeling a fin or keel attached to the body, in the case where there is circulation around the body.

1. Introduction and outline

In this paper, we study the equations of motion for mechanical systems on central extension type Lie groups with nonholonomic constraints, where both the constraints and the kinetic energy are invariant under the left action of the group on itself. Our main motivating example comes from hydrodynamics and consists of a nonholonomic sleigh immersed in a two-dimensional potential flow with circulation.

The Euler–Poincaré–Suslov equations. An *LL system* is a mechanical system on a Lie group G with a kinetic energy Lagrangian and a set of nonholonomic constraints, so that both the Lagrangian and the constraints are left-invariant under the action of G on itself. Due to the invariance under the group action, the dynamics reduce to the Lie algebra $\mathfrak{g}$, or to its dual $\mathfrak{g}^*$ if working with the momentum formulation. The resulting reduced equations are termed the *Euler–Poincaré–Suslov* (EPS) equations [1].

In this paper, we consider EPS equations associated to nonholonomic LL systems for which the underlying Lie group is a *central extension*. We apply the criterion of Jovanović [2] (see also [3]) to obtain necessary and sufficient conditions on the existence of invariant measures for these equations. One of our theoretical results is Theorem 3.1, which states that an EPS system on a central extension $\widehat{G}$ has an invariant measure if and only if the corresponding system on the original Lie group G has an invariant measure.

The hydrodynamic Chaplygin sleigh. Our motivating example of an EPS system on a central extension is given by the motion of a two-dimensional rigid body which moves inside a potential flow with circulation $\kappa \neq 0$, where the nonholonomic constraint precludes motion transversal to the body, modeling, for instance, a very effective keel or fin.

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This model was first considered in the absence of circulation in [4], where it was termed the *hydrodynamic Chaplygin sleigh*. This terminology reflects the fact that in the absence of the fluid, the nonholonomic constraint models the effect of a sharp blade in the classical Chaplygin sleigh problem [5] which prevents the sleigh from moving in the lateral direction. In the presence of the fluid, the constraint can be interpreted as modeling the effect of a very effective keel or fin on the body [4]. Similar models for two-dimensional swimmers have been studied in [6] (see also [7]), while the motion of the hydrodynamic Chaplygin sleigh in the presence of circulation is treated in [8]. However, to the best of our knowledge this is the first time that the geometric nature of the system is elucidated.

The Chaplygin–Lamb equations. When the effect of the keel is ignored, so that there are no nonholonomic constraints, the equations of motion for the hydrodynamic sleigh reduce to the so-called Chaplygin–Lamb equations [9,10].[1] We show that these equations can be viewed in two different, but equivalent ways.

1. As a left-invariant system on the group SE(2) of translations and rotations in the plane, moving under the influence of a *gyroscopic force*. The latter is termed the *Kutta–Zhukovsky force* [11] and models the effect of nonvanishing circulation on the body.
2. As a geodesic system on a *central extension* of SE(2) by $\mathbb{R}^3$ that we denote by $\widehat{G}$, where the extra variables in the $\mathbb{R}^3$-factor describe the circulation. In this way, the Kutta–Zhukovsky force becomes a geometric effect, which is not added explicitly to the system but appears a posteriori as a consequence of how the central extension is constructed.

A classical counterpart of this duality is the description of a particle of charge e moving under the influence of a magnetic field B perpendicular to the plane of motion. As is well known, such a particle may be modeled either as moving under the influence of the Lorentz force, or as a particle moving in the Heisenberg group $\mathbb{R}^2 \times \mathbb{S}^1$ equipped with a group multiplication involving the magnetic field, which appears as a *cocycle* (see [12]). In the example of the hydrodynamic Chaplygin sleigh, the circulation κ plays the role of the charge, and the cocycle will be discussed below.

The nature of the cocycles. In constructing the extension $\widehat{G}$ of SE(2) by $\mathbb{R}^3$, we introduce an $\mathbb{R}^3$-valued two-cocycle $C : \mathfrak{se}(2) \times \mathfrak{se}(2) \to \mathbb{R}^3$ (i.e. a two-cochain which is closed in the sense of Lie algebra cohomology—see below) which can be decomposed on fluid-dynamical grounds as $C = (C_1, C_2)$, where C_1 takes values in $\mathbb{R}$ while C_2 is $\mathbb{R}^2$-valued. As we have pointed out before, each of these cocycles is responsible for the appearance of certain gyroscopic forces in the equations of motion, and we now discuss these forces some more.

The first cocycle, C_1, is "essential" in the sense that it cannot be written as the coboundary (defined below) of a one-cochain $\mathscr{A}$, and we argue that this is a consequence of Kelvin's theorem, which states that circulation is constant. By contrast, the second cocycle C_2 is exact, and we show that it can be "gauged away" by adequately choosing the origin of the body reference frame. From a physical point of view, the cocycle C_2 is associated to the moment generated by the Kutta–Zhukovsky force. While it would have been possible to get rid of C_2, this would complicate the description of the nonholonomic constraint that we discuss below.

Adding nonholonomic constraints. The effect of the keel gives rise to a nonholonomic constraint on the system, which can be viewed as follows: if we affix a frame $\{\mathbf{E}_1, \mathbf{E}_2\}$ to the body, with $\mathbf{E}_1$ aligned with the keel and $\mathbf{E}_2$ perpendicular to it, the effect of the keel is to preclude motion in the $\mathbf{E}_2$-direction, or in other words

$$v_2 = 0, \tag{1.1}$$

where v_2 is the component of the body velocity in the direction of $\mathbf{E}_2$. This is a constraint on the velocities which cannot be integrated to give a relation between the admissible configurations of the body, and is therefore nonholonomic. Just as the kinetic energy, this constraint is left invariant under the action of the central extension $\widehat{G}$ on itself, and therefore gives rise to an EPS system on $\widehat{\mathfrak{g}}$, the Lie algebra of $\widehat{G}$.

Using the geometric structure of the equations, we are able to obtain necessary and sufficient conditions for the existence of an invariant measure (Proposition 4.3). Among other things, we show that the existence of an invariant measure is independent of the circulation.

Outline. The paper is organized as follows. In Section 2 we recall the necessary preliminaries on the theory of central extensions of Lie groups with the viewpoint on their mechanical applications. In Section 3 we consider the EPS equations of LL systems whose underlying Lie group is a central extension and give conditions for the existence of invariant measures in Theorem 3.1.

Section 4 considers the structure of the equations of motion for planar rigid bodies moving on a perfect fluid with circulation. Our contributions are contained in Theorem 4.2 that describes the geometric structure of the most general form of the Chaplygin–Lamb equations and in the construction in Section 4.3 where we show that the reduced equations for the hydrodynamic Chaplygin sleigh with circulation are of EPS type, and where we give conditions for measure preservation. These results rely on the introduction of the group cocycle that defines a central extension of the group SE(2) described above, and that is studied in detail in Section 5.

[1] It is an interesting historic coincidence that the name of Chaplygin is linked both to the development of the Chaplygin–Lamb equations [9] as well as to the nonholonomic Chaplygin sleigh [5].