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Real spectral triples over noncommutative Bieberbach manifolds

Piotr Olczykowski [a], Andrzej Sitarz [b,*,1]

[a] Copernicus Center for Interdisciplinary Studies, Sławkowska 17, 31-016 Kraków, Poland

[b] Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

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ABSTRACT

We classify and construct all real spectral triples over noncommutative Bieberbach manifolds, which are restrictions of irreducible, real, equivariant spectral triples over the noncommutative three-torus. We show that, in the classical case, the constructed geometries correspond exactly to spin structures over Bieberbach manifolds and the Dirac operators constructed for a flat metric.

1. Introduction

Bieberbach manifolds are compact manifolds, which are quotients of the Euclidean space by a free, properly discontinuous and isometric action of a discrete group. The torus is the canonical example of a Bieberbach manifold, however, the first nontrivial example appears in dimension 2 and is a Klein bottle. The case $d = 3$ has already been described in the seminal works of Bieberbach [1,2]. In this paper we work with the dual picture, looking at the suitable algebra of functions on the Bieberbach manifold (and their noncommutative counterparts) in terms of a fixed point subalgebra of the relevant dense subalgebra of the C^* algebra of continuous functions on the three-torus and its corresponding noncommutative deformation $\mathcal{A}(\mathbb{T}^3_\theta)$.

1.1. Noncommutative Bieberbach manifolds

In this section we shall briefly recall the description of three-dimensional noncommutative Bieberbach manifolds as quotients of the three-dimensional noncommutative tori by the action of a finite discrete group. For details we refer to [3], here we present the notation and the results. Out of 10 different Bieberbach three-dimensional manifolds (six orientable, including the three-torus itself and four nonorientable ones) only six have noncommutative counterparts.

* Corresponding author. Tel.: +48 605269245.
E-mail addresses: piotr.olczykowski@uj.edu.pl (P. Olczykowski), andrzej.sitarz@uj.edu.pl (A. Sitarz).

1 Most of this work was carried out at the Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, Warszawa, 00-950 Poland.

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Table 1
The action of finite cyclic groups on $A(T^3_\theta)$.

Name	$\mathbb{Z}_N$	Generator	Action of $\mathbb{Z}_N$ on U, V, W
$B2_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = -U, h \triangleright V = V^*, h \triangleright W = W^*$,
$B3_\theta$	$\mathbb{Z}_3$	$h, h^3 = \mathrm{id}$	$h \triangleright U = e^{\frac{2}{3}\pi i}U, h \triangleright V = e^{-\pi i\theta}V^*W, h \triangleright W = V^*$,
$B4_\theta$	$\mathbb{Z}_4$	$h, h^4 = \mathrm{id}$	$h \triangleright U = iU, h \triangleright V = W, h \triangleright W = V^*$,
$B6_\theta$	$\mathbb{Z}_6$	$h, h^6 = \mathrm{id}$	$h \triangleright U = e^{\frac{1}{3}\pi i}U, h \triangleright V = W, h \triangleright W = e^{-\pi i\theta}V^*W$,
$N1_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = U^*, h \triangleright V = -V, h \triangleright W = W$,
$N2_\theta$	$\mathbb{Z}_2$	$h, h^2 = \mathrm{id}$	$h \triangleright U = U^*, h \triangleright V = -V, h \triangleright W = WU^*$,

Definition 1.1 (*See [3]*)**.** Let $\mathcal{A}(\mathbb{T}^3_\theta)$ be an algebra of smooth elements on a three-dimensional noncommutative torus, which contains the polynomial algebra generated by three unitaries U, V, W satisfying relations,

$$UV = VU, \qquad UW = WU, \qquad WV = e^{2\pi i\theta}VW,$$

where $0 < \theta < 1$ is irrational. We define the algebras of noncommutative Bieberbach manifolds as the fixed point algebras of the following actions of finite groups G on $\mathcal{A}(\mathbb{T}^3_\theta)$, which are combined in Table 1.

We have shown in [3] that the actions of the cyclic groups $\mathbb{Z}_N$, $N = 2, 3, 4, 6$ on the noncommutative three-torus, as given in Table 1is free. The aim of this paper is to study and classify flat (i.e. restricted from flat geometries of the torus $\mathcal{A}(\mathbb{T}^3_\theta)$) real spectral geometries over the orientable noncommutative Bieberbachs.

2. Spectral triples over Bieberbachs

Since each noncommutative Bieberbach algebra is a subalgebra of the noncommutative torus, a restriction of the spectral triple over the latter to the subalgebra, gives a generic spectral triple over a noncommutative Bieberbach manifold, which might be, however, reducible. By restriction of a spectral triple $(\mathcal{A}, \pi, \mathcal{H}, D, J)$ to a subalgebra, $\mathcal{B} \subset \mathcal{A}$, we understand the triple $(\mathcal{B}, \pi, \mathcal{H}', D', J')$ where π' is the restriction of π to $\mathcal{B}$, $\mathcal{H}' \subset \mathcal{H}$ is a subspace invariant under the action of $\mathcal{B}$, D and J, so that D', J' are their restrictions to $\mathcal{H}'$ (note that in the even case this must apply also to γ).

In what follows we shall show that, in fact, each spectral triple over Bieberbach is a restriction of a spectral triple over the torus, first showing that each spectral triple over Bieberbach can be lifted to a noncommutative torus.

2.1. The lift and the restriction of spectral triples

Lemma 2.1. *Let* $(\mathrm{BN}_\Theta, \mathcal{H}, J, D)$ *be a real spectral triple over a noncommutative Bieberbach manifold* BN_θ*. Then, there exists a spectral triple over a three-torus, such that this triple is its restriction.*

Proof. In [3] we showed that the crossed product algebra $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$ is isomorphic to the matrix algebra of the noncommutative Bieberbach manifold algebra:

$$\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N \sim \mathrm{BN}_\theta \otimes M_N(\mathbb{C}).$$

First, let us recall that any spectral triple $\mathcal{A}, \mathcal{H}, D, J$ could be lifted to a spectral triple over $\mathcal{A} \otimes M_n(\mathbb{C})$. Indeed, if we take $\mathcal{H}' = \mathcal{H} \otimes M_n(\mathbb{C})$ with the natural representation $\pi'(a \otimes m)(h \otimes M) = \pi(a)h \otimes mM$, the diagonal Dirac operator and $J'(h \otimes M) = Jh \otimes UM^\dagger U^\dagger$, for an arbitrary unitary $U \in M_n(\mathbb{C})$ it is easy to see that we obtain again a real spectral triple. Applying this to the case of BN_θ, and identifying $\mathrm{BN}_\theta \otimes M_N(\mathbb{C})$ using the above isomorphism we obtain a spectral triple over $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$. As $\mathcal{A}(\mathbb{T}^3_\theta)$ is a subalgebra of $\mathcal{A}(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$ by restriction we obtain, in turn, a spectral triple over a three-dimensional noncommutative torus. In fact, it is easy to see that we obtain a spectral triple, which is equivariant with respect to the action of a $\mathbb{Z}_N$ group. Clearly, the fact that we have a representation of the crossed product algebra is just a rephrasing of the fact that we have a $\mathbb{Z}_N$-equivariant representation of $\mathcal{A}(\mathbb{T}^3_\theta)$. By definition, the Dirac operator lifted from the spectral triple over BN_θ commutes with the group elements which are identified as matrices in $M_N(\mathbb{C})$. A little care is required to show that the lift of J would properly commute with the generator of $\mathbb{Z}_N$. However, since the lift of J involves a matrix U, it is sufficient to use a matrix, which provides a unitary equivalence between the generator h and its inverse h^{-1} of $\mathbb{Z}_N$ in $M_N(\mathbb{C})$. Simple computation shows that the following U, $U_{00} = 1$, $U_{kl} = \delta_{k,N-l}$ for $k, l = 1, \ldots, N-1$ is the one providing the equivalence.

Hence, by this construction, we obtain a real, $\mathbb{Z}_N$-equivariant spectral triple over $\mathcal{A}(\mathbb{T}^3_\theta)$. It is easy to see that the original spectral triple over BN_θ is a restriction of the constructed spectral triple by taking the invariant subalgebra of $\mathcal{A}(\mathbb{T}^3_\theta)$, the $\mathbb{Z}_N$-invariant subspace of $\mathcal{H}$ and the restriction of D and J. □

Remark 2.2. The procedure described above does not necessarily provide the *canonical* (equivariant) Dirac operator over $\mathcal{A}(\mathbb{T}^3_\theta)$. Indeed, even a simple example of $\mathcal{A}(\mathbb{T}^1)$ shows that the lifted Dirac operator differs from the fully equivariant one by a bounded term. There may exist, however, a fully equivariant triple, so that its restriction is the same triple we started with.