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The Björling problem for timelike surfaces in $\mathbb{R}^4_2$

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ABSTRACT

We solve the Björling problem for timelike surfaces in $\mathbb{R}^4_2$ constructing a special normal frame and a split-complex representation formula. We use this solution to construct new examples of timelike minimal surfaces and to define a notion of symmetry appropriate to the indefinite setting. We also establish results describing various symmetries for this kind of surface.

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1. Introduction

The Björling problem has been investigated in various settings over a long period of time and has yielded many interesting results. In 1844 Björling asked about the construction of a minimal surface in $\mathbb{R}^3$ containing a prescribed analytic strip. Schwarz gave an explicit solution to this problem in 1890. Later, the Björling problem was considered in other ambient spaces, including some with indefinite metrics. For instance in [1], they solved the problem for spacelike surfaces in $\mathbb{L}^3$, while in [2] the solution to the Björling problem is established for timelike surfaces in $\mathbb{L}^3$. For codimensions bigger than one, there is [3] and, more recently the paper of Asperti and Vilhena, [4], which studies the Björling problem for spacelike surfaces in $\mathbb{L}^4$. Other references, using different ambient spaces, are [5,6,1,7,8].

In this paper we solve the Börling problem for timelike surfaces in $\mathbb{R}^4_2$. To prove our results, we use the split-complex numbers (which are also known as the para-complex, double, Lorentz or hyperbolic numbers), henceforth denoted by $\mathbb{C}'$. These numbers are particularly useful when studying timelike surfaces. In fact, they allow us to import some of the formalisms from complex variables used in the study of minimal surfaces. One of our long-term goals is to see how far the analogy with complex analysis can be taken. This analogy is not perfect, because the split-complex numbers have zero divisors and, so, for example, $1/z$ is singular on a pair of intersecting lines, not one point.

We chose the ambient space $\mathbb{R}^4_2$ because it is the direct sum of two copies of the split-complex numbers. Using this fact, we find a convenient local normal frame to describe the Gauss map of the immersion, which we then utilize to find the split-complex Björling representation formula for minimal timelike surfaces in $\mathbb{R}^4_2$. So as a first consequence, we recover the split-complex representation formula of the Björling problem for minimal timelike surfaces in $\mathbb{L}^3$. In particular, to obtain

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our results, we identify the $G^-_{2,4}$, the Grassmannian of oriented timelike planes in $\mathbb{R}^4_2$, with the quadric $Q^2_1 = \{[z] \in \mathbb{C}'\mathbb{P}^{3-}_2 \mid (z, z) = 0\}$, where $(\,,\,)$ is a symmetric bilinear product associated to an indefinite Hermitian structure on $\mathbb{C}'^{\,4}$. The projective space $\mathbb{C}'\mathbb{P}^{3-}_2$ is the quotient space of H^7_3 modulo the action by $H^1 = \{\lambda \in \mathbb{C}' | \lambda\bar{\lambda} = 1\}$. To give a unique solution to the Björling problem we need only to consider the timelike and spacelike curves with a normal field. In other words, we must exclude lightlike curves, which are related to the characteristic curves for the determining partial differential equation, just as occurs in [2]. Thus we set up and study two Björling problems, namely, the timelike and spacelike Björling problems.

As part of our exploration of split-complex analysis, we prove the split-complex version of Schwarz reflection, and then we introduce the notion of k-subspace of symmetry for timelike surfaces in $\mathbb{R}^4_2$, including degenerate and non-degenerate subspaces. In fact, this definition is easily extended to any $\mathbb{R}^n_j$. As noted below, our definition corrects an inconsistency in [4]. Using these notions and our split-complex Björling representation formulas, we are able to describe three types of symmetries for minimal timelike surfaces in $\mathbb{R}^4_2$ with respect to non-degenerate k-subspaces. For the degenerate subspace case of type −0, we also obtain some information, assuming symmetry with respect to 2-plane of that type.

2. Preliminaries

Let $\mathbb{R}^4_2$ be $\mathbb{R}^4$ with the indefinite inner product

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4,$$

which is our ambient space.

The definition of the cross-product in a three-dimensional vector space with metric $b(u, v)$ is defined, in [9], to be:

$$b(u \times_b v, z) = \begin{vmatrix} z \\ u \\ v \end{vmatrix} = \det \begin{pmatrix} z \\ u \\ v \end{pmatrix}. \tag{1}$$

So, for example, in the space where $b((u_2, u_3, u_4), (v_2, v_3, v_4)) = -u_2v_2 + u_3v_3 + u_4v_4$, which would occur when we drop the first coordinate of elements in $\mathbb{R}^4_2$, one obtains

$$(u_2, u_3, u_4) \times_b (v_2, v_3, v_4) = \begin{vmatrix} -i & j & k \\ u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \end{vmatrix} = (u_4v_3 - u_3v_4, u_4v_2 - u_2v_4, u_2v_3 - v_2u_3). \tag{2}$$

For $\mathbb{R}^4_2$ we define the cross product $\boxtimes$ by

$$\langle \boxtimes(u, v, w), x\rangle = \begin{vmatrix} x \\ u \\ v \\ w \end{vmatrix}, \tag{3}$$

where $u, v, w, x \in \mathbb{R}^4_2$.

With this definition:

$$\boxtimes(u, v, w) = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4 \tag{4}$$

where

$$\alpha = -\begin{vmatrix} u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \\ w_2 & w_3 & w_4 \end{vmatrix}, \quad \beta = \begin{vmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ w_1 & w_3 & w_4 \end{vmatrix}, \quad \gamma = \begin{vmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{vmatrix}, \quad \delta = -\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{5}$$

It then follows that $\langle \boxtimes(u, v, w), u\rangle = 0 = \langle \boxtimes(u, v, w), v\rangle = \langle \boxtimes(u, v, w), w\rangle$,

$$\boxtimes(u, v, e_1) = -\breve{u} \times \breve{v} \tag{6}$$

$$\boxtimes(u, v, e_2) = \breve{u} \times \breve{v} \tag{7}$$

$$\boxtimes(u, v, e_3) = -\breve{u} \times \breve{v} \tag{8}$$

$$\boxtimes(u, v, e_4) = \breve{u} \times \breve{v} \tag{9}$$

where $\breve{u}$ means dropping the first coordinate in the first case, the second coordinate in the second case, etc. and the cross product is taken in the three dimensional space with the metric inherited from $\mathbb{R}^4_2$.

We can see that

$$\langle \boxtimes(u_1, u_2, u_3), \boxtimes(v_1, v_2, v_3)\rangle = \left|\langle u_i, v_j\rangle\right|, \quad 1 \le i, j \le 3,\ u_i, v_j \in \mathbb{R}^4_2. \tag{10}$$

We denote the span of a set of vectors $\{x_1, \dots, x_n\}$ in $\mathbb{R}^4_2$ by $[x_1, \dots, x_n]$.

As noted above, the split-complex numbers $\mathbb{C}'$ are helpful when studying timelike surfaces.

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