Journal of Geometry and Physics 68 (2013) 81–102

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On discrete differential geometry in twistor space

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ARTICLE INFO

Article history:
Received 29 March 2011
Received in revised form 13 February 2013
Accepted 19 February 2013
Available online 26 February 2013

Keywords:
Discrete differential geometry
Discrete integrable systems
Plücker quadric
Conformal differential geometry
Quaternionic projective space
Cross-ratio

ABSTRACT

In this paper we introduce a discrete integrable system generalizing the discrete (real) cross-ratio system in S^4 to complex values of a generalized cross-ratio by considering S^4 as a real section of the complex Plücker quadric, realized as the space of two-spheres in S^4. We develop the geometry of the Plücker quadric by examining the novel contact properties of two-spheres in S^4, generalizing classical Lie geometry in S^3. Discrete differential geometry aims to develop discrete equivalents of the geometric notions and methods of classical differential geometry. We define discrete principal contact element nets for the Plücker quadric and prove several elementary results. Employing a second real structure, we show that these results generalize previous results by Bobenko and Suris (2007) [18] on discrete differential geometry in the Lie quadric.

1. Introduction

The study of special surfaces in three-dimensional space has been a topic of interest going back to the foundations of differential geometry. The subject encompasses surfaces in Euclidean space with familiar properties e.g. constant mean curvature, constant Gaussian curvature and surfaces in classical affine, Möbius (conformal), and projective geometries and also surfaces in unfashionable [1] geometries such as Lie geometry [2]. This view was summarized at the beginning of the 20th century in the compendium of Blaschke [3]. Parameterizations of surfaces in space are determined by solutions of partial differential equations. Then, integrability conditions on these equations for special surfaces determine completely integrable partial differential equations. This was an essential component of classical results and was rediscovered in the latter part of 20th century with the redevelopment of "soliton" geometry [4].

It is a characteristic of the smooth theory that surfaces in 3-space can often be found in families related by fundamental transformations such as the Ribaucour transformation [5]. Under the rubric of "Discrete Differential Geometry", recent work has discretized classical surface theory by modeling the special differential equations of surface theory with partial *difference* equations [6,7]. This has required the development of a discrete theory of "integrability" [8] defined by modeling the permutability relations (e.g. Bianchi permutability) between transformations of smooth solutions of integrable systems as a "consistency" condition in the discrete case [6]. Then, one is lead to the viewpoint that the geometry of the fundamental transformations and the geometry of the *discrete* surface are the same [9].

Given a parameterization of a surface in space with coordinates (u, v), a system of coordinate curves defined by $u =$ const. and $v =$ const. determines a "net" on the surface. The classical approach to special surfaces starts with the geometry of these coordinate curves. In particular, "conjugate nets", are parameterizations of surfaces in space defined by the property that embedded tangent vectors at each point on a coordinate curve stay in the tangent plane under motions of the curve in the transverse coordinate direction. This is a concept of projective differential geometry and in an affine chart is equivalent to the property that the mixed partial derivative of the parameterization is, at each point, spanned by tangent vectors [10]. Conjugate nets parameterize constant mean curvature, constant Gauss curvature and isothermic surfaces, among others [6].

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http://dx.doi.org/10.1016/j.geomphys.2013.02.008

With this in mind, we consider the discretization of a parameterization on a surface by a "discrete net" of discrete coordinate curves: a map $f : \mathbb{Z}^2 \to \mathbb{RP}^n$. Discrete surfaces in the classical geometries are then obtained by quadratic constraints on the image of the discrete net in $\mathbb{RP}^n$ [11], following Klein's projective model [12]. Thus, discrete conformal differential geometry, in the three-dimensional case, considers discrete nets in $\mathbb{RP}^4$ constrained to lie in the light-cone of Lorentzian metric $\mathbb{R}^5$. Discrete nets are determined by solutions of systems of partial difference equations. In particular, the difference equations defining discrete conjugate nets [13] are determined by requiring that the image of each face in $\mathbb{Z}^2$ lies in a projective plane. Discrete conjugate nets form the fundamental example of a discrete integrable system defining a discrete surface as solutions of the difference equation:

$$\Delta_i \Delta_j f = a_{ji} \Delta_i f + b_{ij} \Delta_j f + c_{ij} f,$$

where i, j vary over the lattice coordinates of $\mathbb{Z}^2$ [6]. Results showing convergence from discrete to smooth solutions of the Laplace equation indicate that the discrete and smooth surfaces are closely related [14].

Just as Euclidean geometry is a subgeometry of Möbius geometry, Riemannian geometry is generalized by conformal (Möbius) differential geometry [15]. Möbius geometry is a subgeometry of projective geometry so that, in conformal differential geometry, parameterizations are coordinate maps from a surface into projective space. Isothermal coordinates, parameterizations compatible with the conformal structure of a surface, are employed as a natural construction. Curvature-line coordinates, described typically in terms of the Riemannian principal curvatures end up being conformally invariant. Thus, in conformal differential geometry it is natural to study isothermic surfaces: surfaces parameterized by curvature line coordinates compatible with the conformal structure of the surface.

As the notion of a tangent space embedded at each point of the surface in ambient 3-space is not conformally invariant it is natural to consider the conformally invariant "mean curvature sphere" [16] of a surface. This is a map assigning to each point of the surface in space a tangent sphere with radius given by the inverse of the mean curvature. Hence, the mean curvature sphere is properly a map from the surface into the space of two-spheres: a surface made of spheres. Lie geometry, introduced by Sophus Lie in his doctoral dissertation under Plücker (and in collaboration with Felix Klein [12]) is the geometry of oriented hyperspheres in Möbius geometry. In Klein's projective model, Lie geometry is the geometry of the Lie quadric, a signature $(2, 4)$ real projective quadric. The set of null projective lines in the Lie quadric are an elementary example of contact geometry, with each null line corresponding to the one-parameter family of spheres tangent with a given orientation at a fixed point in space. As the elementary invariant sets of Möbius geometry are spheres, there is a natural relationship between conformal geometry and Lie geometry. Indeed, Möbius geometry is contained as a subgeometry of Lie geometry with the 3-sphere contained in the Lie quadric as the set of 2-spheres of zero radius. The Möbius group is the subgroup of symmetries preserving these point-spheres, "point transformations" in the general group of symmetries of the Lie quadric [17]. A surface in space naturally induces a map into the space of contact lines in the Lie quadric determined by the set of tangent spheres at each point of the surface. Each contact line contains a point corresponding to a sphere of 0-radius, a point in S^3. Thus, a map into the space of contact lines which satisfies the Legendre condition determines a surface in S^3 [2].

In the survey article of Bobenko and Suris [18] the theory of discrete differential geometry is extended to contact lines in Lie geometry. Discrete "principal contact element nets" model the family of contact lines associated to a surface parameterized in curvature line coordinates. The discrete analog of curvature line coordinates for surfaces in space are "circular nets", maps $\mathbb{Z}^2 \to S^3$ where the points of each elementary quadrilateral of the lattice lie on a circle. Then, each contact line contains a representative corresponding to a point in S^3 so that the set of such point-spheres determines a circular net in S^3 [6]. In the light-cone model of S^3, a circle is given by the intersection of a projective 2-plane with $S^3 \subset \mathbb{RP}^4$. Thus, circular nets in S^3 are conjugate nets in $\mathbb{RP}^4$ subject to the constraint determined by the quadratic form of the light-cone [11].

Circular nets in S^2 may be considered as a discretization of the Gauss map for a discrete surface [14,19]. Identifying S^2 with $\mathbb{CP}^1$, the cross-ratio of four circular points $[q_1, q_2, q_3, q_4] \in \mathbb{R}$. Thus, circular nets in S^2 are solutions of the "cross ratio system", a set of partial difference equations given by determining the cross-ratio for each face of the map on $\mathbb{Z}^2$. Given three points $\{q_1, q_2, q_3\} \in \mathbb{CP}^1$, a complex number λ determines a unique fourth point q_4 so that the four points have cross-ratio given by λ. This cross-ratio system extends naturally to complex values and general discrete nets in S^2. Thus in S^2, the complex cross-ratio defines a master system of which circular nets are special solutions.

Let c be a discrete curve given as a map $c : \mathbb{Z} \to \mathbb{C} \subset \mathbb{CP}^1$, then a choice of an initial point $c^+(0)$ defines the first iteration of the discrete evolution of c by the formula:

$$[c(1), c(0), c^+(0), c^+(1)] = \lambda.$$

Given $\lambda \in \mathbb{C}$ as a parameter, $c^+(1) = M_0(\lambda)c^+(0)$, where $M_0(\lambda)$ is a Möbius transformation of S^2. If $\lambda \in \mathbb{R}$, then the points $\{c(1), c(0), c^+(0), c^+(1)\}$ are concircular. Iterating this procedure, specifying $\lambda : \mathbb{Z}^2 \to \mathbb{R}$ determines a circular net in S^2. Thus, a discrete net may be viewed as the discrete evolution of a discrete coordinate curve. If c is a closed curve, that is $c : \mathbb{Z} \to \mathbb{H}$ is periodic, then iterating $M_k(\lambda)$ around c one is led to the holonomy problem

$$M_n(\lambda) \dots M_1(\lambda)M_0(\lambda)c^+(0) = c^+(0) \tag{1.1}$$

given by the closing condition around c. Thus, the eigenlines of the holonomy problem determine initial conditions for the evolution of a closed curve c.