

The Arnold cat map, the Ulam method and time reversal

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ABSTRACT

We study the properties of the Arnold cat map on a torus with several periodic sections using the Ulam method. This approach generates a Markov chain with the Ulam matrix approximant. We study numerically the spectrum and eigenstates of this matrix showing their relation with the Fokker–Planck relaxation and the Kolmogorov–Sinai entropy. We show that, in the frame of the Ulam method, the time reversal property of the map is preserved only on a short Ulam time which grows only logarithmically with the matrix size. Parallels with the evolution in a regime of quantum chaos are also discussed.

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1. Introduction

The Arnold cat map [1] is the cornerstone model of the classical dynamical chaos [2–4]. This symplectic map belongs to the class of Anosov systems, has the positive Kolmogorov–Sinai entropy $h \approx 0.96$ and is fully chaotic [4]. The map has the form

$$\bar{p} = p + x \pmod{L}, \quad \bar{x} = x + \bar{p} \pmod{1}. \quad (1)$$

Here the first equation can be seen as a kick which changes the momentum p of a particle on a torus while the second one corresponds to a free phase rotation in the interval $-0.5 \leq x < 0.5$; bars mark the new values of canonical variables (x, p) . The map dynamics takes place on a torus of integer length L in the p direction with $-L/2 < p \leq L/2$. The usual case of the Arnold cat map corresponds to $L = 1$ but it is also possible to study the chaotic properties of the map on a torus of longer integer size $L > 1$ as it has been discussed e.g. in [5]. For $L \gg 1$, the spreading in p is characterized by a diffusive process described by the Fokker–Planck equation:

$$\partial w(p, t) / \partial t = D / 2 \partial^2 w(p, t) / \partial^2 p, \quad (2)$$

where the diffusion coefficient $D \approx \langle x^2 \rangle = 1/12 w(p, t)$ is a probability distribution over momentum and t is the integer time measured in number of iterations. As a result, for times

$t \gg L^2/D$, the distribution converges to the ergodic equilibrium with a homogeneous density in the plane (x, p) . The exponential convergence to the equilibrium state is determined by the second eigenvalue λ_2 of evolution (2) on one map iteration with $|\lambda| = \exp(-\Gamma_D) < 1$ and the convergence rate

$$\Gamma_D = 2\pi^2 D / L^2 \approx 1.6449 / L^2; \quad (3)$$

the first eigenvalue is $\lambda_1 = 1$.

The dynamical equations (1) are reversible in time, e.g. in the middle of free rotation, but, due to chaos and exponential instability of motion, small round-off errors break time reversal leading to an irreversible relaxation to the ergodic equilibrium [5].

In this work, we investigate the transition from dynamical behavior to statistical description using the Ulam method proposed in 1960 [6]. According to this method, the whole phase space is covered by equidistant lattice ($N = N_p \times N_x$ in our case). Then the transition probabilities from cell to cell are determined by propagating a large number of trajectories N_{tr} from one initial cell j to all other cells i after one iteration of the map (we used here $N_{tr} = 10^5$). In this way, we generate the Markov chain [7] with a transition matrix $S_{ij} = N_{ij} / N_{tr}$, where N_{ij} is the number of trajectories arrived from cell j to cell i . By construction, we have $\sum_{i=1}^N S_{ij} = 1$ and thus the matrix S belongs to the class of Perron–Frobenius operators [1,2,8]. It is proven that for hyperbolic maps in one and higher dimensions the Ulam method gives a convergence to the spectrum of continuous system in the limit of small cell size [9–11]. At the same time it is known that in certain cases the Ulam method gives significant modifications of the spectrum compared to the case of the continuous Perron–Frobenius operators [10]. Indeed, for Hamiltonian maps with divided phase space, the spectrum is completely

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modified (see discussions in [12,13]) due to penetration of trajectories inside stability islands. From a physical view point the discretization corresponds to an effective noise in canonical variables which amplitude is equal to the cell size. Since an arbitrary small noise gives propagation of trajectories inside stability islands [4] the spectrum of the Ulam matrix approximant of size N in such a case differs from its continuous limit. A generalization of the Ulam method, based on one ergodic trajectory, allows to obtain a convergent spectrum for dynamics on a chaotic component [13].

The majority of numerical studies with the Ulam method has been done for one-dimensional maps (see e.g. [14–16]) but recently the studies were extended to the two-dimensional maps (see e.g. [12,13,17,18]). For example, a geometrical description of coherent flow structures and their invariant manifolds has been developed in [17], the fractal Weyl law for dissipative maps was discussed in [18]. In a certain respect the interest to such studies was generated by similarities between properties of the Ulam matrix approximant for dynamical maps, which can be viewed as the Ulam networks, and the Google matrix of the World Wide Web as it is discussed in [12,16]. For 2D dissipative maps it was found that the spectrum is characterized by the fractal Weyl law [12,18].

In a difference from the previous studies of the Ulam method in 2D maps here we choose the Arnold cat map on a torus of size L since it is fully chaotic, it has well defined diffusive relaxation to the ergodic state at large L , and it is time reversible. Thus the aim of this work is to understand the interplay of all these features in the frame of the Ulam method and the finite size Markov chain with the Ulam matrix approximant S generated by this method. The previous studies of toral automorphisms by the Ulam method performed in [13,17] were done for the regimes where the diffusion time scale was comparable with the time scale given by the inverse Lyapunov exponent. In the case considered here we have the diffusion time being significantly larger than the Lyapunov time scale, thus our studies are done in the regime which satisfies the Bogolubov criterion on time scales separation required for a statistical description.

The paper is composed as follows: in Section 2 we describe the properties of spectrum and eigenstates of the matrix S , the features of time reversal are analyzed in Section 3 and discussion of the results is presented in Section 4.

2. Spectrum and eigenstates of the Ulam matrix approximant

The complex eigenvalues λ_i and right eigenvectors ψ_i of the Ulam matrix approximant S satisfy the equation $S\psi_i = \lambda_i\psi_i$ and are determined numerically by direct diagonalization. In agreement with the Perron–Frobenius theorem [8] the maximal eigenvalue is $\lambda_1 = 1$ with the corresponding eigenstate being real, nonnegative and homogeneously distributed over the whole phase space.

The global distributions of eigenvalues λ_i in the complex plane are shown in Fig. 1 for even and odd number of cells. Usually we keep $N_p = LN_x$ to have exactly the same amount of cells in each of L sections of the continuous map. The results show that 2D distributions are different for even and prime values of N_x (see Fig. 1). For the even case λ -values are homogeneous inside a circle of a certain radius. For the odd case the distribution has a form of a ring without eigenvalues at $|\lambda| \approx 0$ (or with a smaller density at zero). The arithmetic properties of the number of cells N_x and $N_p = LN_x$ play a visible role. Thus for $N_x = 47$ we have a formation of star with 16 star rays while for $N_x = 43$ there are 44 rays which are much less visible (for $N_x = 37$ we obtain a similar type of distribution with 38 rays). We obtain a similar type of ring spectrum also for $N_x = 51$ (bottom right panel in Fig. 1). In the case when both N_x and N_p are primes, e.g. $N_x = 47$, $N_p = 191$, (and hence we have only approximate relation $N_p \approx LN_x$) the

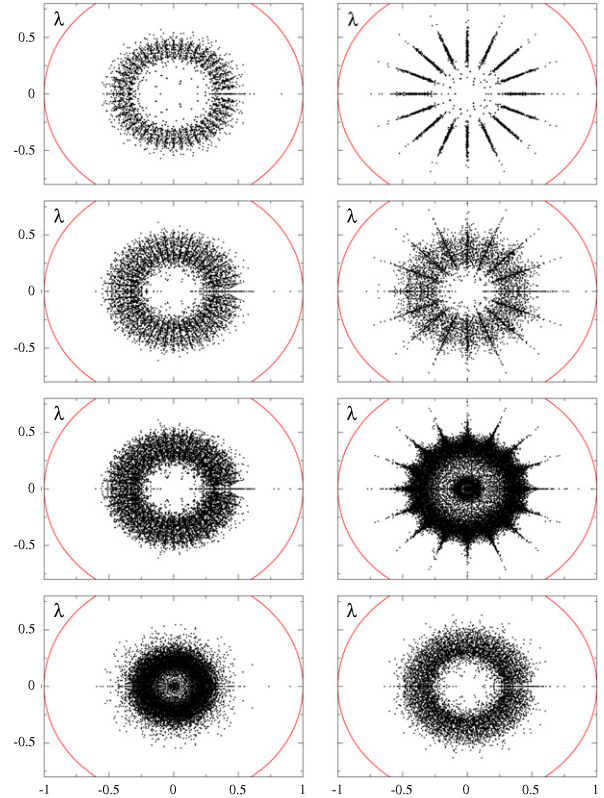


Fig. 1. Complex spectrum λ of the Ulam matrix approximant S for the Arnold cat map. Top three rows: right column has $N_x = 47$ and $L = 3, 4, 8$ (from top to bottom); left column has $N_x = 43$ and $L = 3, 4, 8$ (from top to bottom). Bottom row is for $L = 4$ with $N_x = 50$ (left) and $N_x = 51$ (right). The total matrix size is $N = N_p N_x$ with $N_p = LN_x$ and all eigenvalues are shown for each panel. Unit circle is shown in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

visibility of rays also decreases (data not shown). It is clear that the arithmetic properties of the discretized lattice play an important role at small lattice sizes. However, we argue that they will be not so important in the limit of large lattice size.

Indeed, we expect that in the limit of large N_x and $N_p = LN_x$ with fixed L the distribution will converge to a limiting one in agreement with the spirit of mathematical theorems about the convergence of Ulam matrix approximant for fully chaotic maps [9–11]. A confirmation of this is seen in Fig. 2 where the density distributions of eigenvalues $dW/d\gamma$ are shown as a function of the relaxation rate $\gamma = -2 \ln |\lambda|$. Indeed, the density is essentially size independent showing two distinct distributions for even and odd values of N_x . We suppose that this difference between two cases can be related to the effect of discretization on the continuous map symmetry $x \rightarrow -x$. The third type of size independent distribution appears in the case of prime values of N_x and $N_p \approx LN_x$ (see Fig. 2) but in this case the difference should be attributed to the fact that this discretization does not preserve exactly L identical classical segments of the continuous map.

The maximum of the distribution $dW/d\gamma$ is located approximately at $\gamma = 2 \approx 2h$ corresponding to the value of the Kolmogorov–Sinai entropy h . Thus these γ values describe the process of exponential divergence of nearby trajectories and are related to the exponential correlations decay generated by chaotic dynamics. In addition to these values $\gamma \sim 1$ there is also the value of $\lambda_2 = \exp(-\gamma_2/2)$ which is positive and is very close to the unit value $\lambda_1 = 1$. It corresponds to the second eigenvalue of the Fokker–Planck equation describing diffusive relaxation to the ergodic steady-state. Indeed, the dependence of the gap $\Delta = 1 - \lambda_2$, shown in Fig. 3, is well in agreement with the dependence (3): a

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