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Nonlinear-damping continuation of the nonlinear Schrödinger equation — A numerical study

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ABSTRACT

We study the nonlinear-damping continuation of singular solutions of the critical and supercritical NLS. Our simulations suggest that for generic initial conditions that lead to collapse in the undamped NLS, the solution of the weakly-damped NLS

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1,$$

is highly asymmetric with respect to the singularity time, and the post-collapse defocusing velocity of the singular core goes to infinity as the damping coefficient δ goes to zero. In the special case of the minimal-power blowup solutions of the critical NLS, the continuation is a minimal-power solution with a higher (but finite) defocusing velocity, whose magnitude increases monotonically with the nonlinear damping exponent q.

1. Introduction

The nonlinear Schrödinger equation (NLS)

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi = 0, \qquad \psi_0(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^1, \quad (1)$$

where $\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $\Delta = \partial_{x_1x_1} + \cdots \partial_{x_dx_d}$, is one of the canonical nonlinear equations in physics, arising in various fields such as nonlinear optics, plasma physics, Bose–Einstein condensates (BECs), and surface waves. When $(p-1)d < 4$, the NLS is called subcritical. In that case, all H^1 solutions exist globally. In contrast, both the critical NLS $(p-1)d = 4$ and the supercritical NLS $(p-1)d > 4$ admit singular solutions. Since physical quantities do not become singular, this implies that some of the terms that were neglected in the derivation of the NLS, become important near the singularity.

The continuation of NLS solutions beyond the singularity has been an open question for many years. In 1992, Merle [1] presented a continuation of the explicit blowup solutions $\psi_{\text{explicit},\alpha}$ of the critical NLS, see (9), which is based on slightly reducing the power (L^2 norm) of the initial condition. This continuation has two key properties:

1. *Property* 1: The solution is symmetric with respect to the singularity time T_c.
2. *Property* 2: After the singularity, the solution can only be determined up to multiplication by a constant phase term $e^{i\theta}$.

More recently, Merle et al. [2] have generalized this continuation result to Bourgain–Wang singular solutions [3]. Note, however, that both the explicit solutions $\psi_{\text{explicit},\alpha}$ and the Bourgain–Wang solutions are unstable.

In [4], Merle presented a different continuation, which is based on the addition of nonlinear saturation. Merle showed that, generically, as the nonlinear saturation coefficient goes to zero, the limiting solution beyond T_c can be decomposed into two components: a δ-function singular core that extends for $T_c \le t \le T^0$, and a regular component elsewhere.

In [5], Tao proved the global existence and uniqueness in the semi Strichartz class for solutions of the critical NLS. Intuitively, these solutions are formed by solving the equation in the Strichartz class whenever possible, and deleting any power that escapes to spatial or frequency infinity when the solution leaves the Strichartz class. These solutions, however, do not depend continuously on the initial conditions, and are thus not a well-posed class of solutions. Recently, Stinis [6] studied numerically the continuation of singular NLS solutions using the t-model approach.

In [7] we analyzed asymptotically and numerically four potential continuations of singular NLS solutions: (1) a sub-threshold power continuation, (2) a shrinking-hole continuation for ring-type solutions, (3) a vanishing nonlinear-damping continuation, and (4) a complex Ginzburg–Landau (CGL) continuation. Our main findings were as follows:

1. The non-uniqueness of the phase of the singular core beyond the singularity (Property 2) is a universal feature of NLS continuations.

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2. The symmetry with respect to the singularity time (Property 1) holds if the continuation model is time reversible and if it leads to a point singularity (i.e., if it defocuses for $t > T_c$). Therefore, it is a non-generic feature.

Recently, the post-collapse loss-of-phase phenomenon was demonstrated experimentally for intense laser beams propagating in water [8].

In this paper we further study the effect of small nonlinear-damping in the NLS

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1. \quad (2)$$

The addition of small nonlinear-damping is physical. Indeed, in nonlinear optics, experiments suggest that arrest of collapse is related to plasma formation, and nonlinear damping is used as phenomenological model for multi-photon absorption by plasma. In BEC, a quintic nonlinear damping term corresponds to losses from condensate due to three-body inelastic recombinations [9]. In addition, the nonlinear-damping term appears in the complex-Ginzburg–Landau (CGL) equation, which arises in a model of chemical turbulence, Poiseuille flow, Rayleigh–Bérnard convection, Taylor–Couette flow, and superconductivity.

In [7] we analyzed the continuation of the critical NLS with a vanishing critical nonlinear damping, i.e., Eq. (2) with $p = q = 1 + 4/d$. Since the NLS (2) is not time reversible, its solutions are asymmetric with respect to the time $T_{\text{arrest}}^{(\delta)}$ at which the collapse is arrested. In particular, in the limit $\delta_n \to 0+$, the continuation of $\psi_{\text{explicit},\alpha}(t,r)$ is $e^{i\theta}\psi^*_{\text{explicit},\kappa\alpha}(2T_c - t, r)$, where $\kappa \approx 1.614$. Hence, the defocusing velocity $\kappa\alpha$ is higher then the focusing velocity α. When the initial condition leads to a log–log collapse in the undamped critical NLS, asymptotic analysis and numerical simulations suggest that the singular core expands beyond the singularity at a velocity that goes to infinity as $\delta \to 0+$.

The question that we address in this study is whether and how the results of [7] for $q = p = 1 + 4/d$ will change in the following cases:

1. The critical NLS with a supercritical damping exponent (i.e., $q > p = 1 + 4/d$).
2. The supercritical NLS with $q \geq p > 1 + 4/d$.

The paper is organized as follows. In Section 2 we provide a short review of NLS theory. In Section 3 we review previous rigorous, asymptotic, and numerical results on the effect of damping in the NLS. In Section 4 we show numerically that in the supercritical NLS, the nonlinear damping exponent q has to be strictly higher than the nonlinearity exponent p, in order to arrest the collapse. This is different from the critical case, where collapse is arrested for $q \geq p$. In Section 5 we show that solutions of the supercritical NLS with a small nonlinear damping are asymmetric with respect to the arrest-of-collapse time $T_{\text{arrest}}^{(\delta)}$, and that the post-collapse defocusing velocity of the singular core goes to infinity as the damping coefficient δ goes to zero. In Section 6 we obtain similar results for the critical NLS with generic initial conditions that lead to a log–log collapse. In the special case of the minimal-power explicit blowup solution $\psi_{\text{explicit},\alpha}(t,r)$ of the critical NLS, however, the continuation beyond the singularity is also defined for $q < p$, and is given by $e^{i\theta}\psi^*_{\text{explicit},\kappa(q)\alpha}(2T_c - t, r)$, where $\kappa(q)$ increases monotonically with q. Final remarks are given in Section 7.

Overall, the qualitative effect of small nonlinear damping on the collapse is the same in the critical and the supercritical NLS. One difference is that in the critical case collapse is arrested for $q \geq p$, whereas in the supercritical case collapse is only arrested for $q > p$. Another difference is that the distance between the damped solution around $T_{\text{arrest}}^{(\delta)}$ and the asymptotic profile of the undamped NLS is small in the critical case, but large in the supercritical case. Surprisingly, in the latter case, the profile near $T_{\text{arrest}}^{(\delta)}$ appears to be given by a rescaled supercritical standing wave.

2. Review of NLS theory

The NLS (1) has two important conservation laws: *Power* conservation[1]

$$P(t) \equiv P(0), \qquad P(t) = \int |\psi|^2 d\mathbf{x},$$

and *Hamiltonian* conservation

$$H(t) \equiv H(0), \qquad H(t) = \int |\nabla\psi|^2 d\mathbf{x} - \frac{2}{p+1}\int |\psi|^{p+1} d\mathbf{x}. \quad (3)$$

The NLS (1) admits the waveguide solutions $\psi = e^{it}R(r)$, where $r = |\mathbf{x}|$, and R is the solution of

$$R''(r) + \frac{d-1}{r}R' - R + R^p = 0, \qquad R'(0) = 0, \qquad R(\infty) = 0. \quad (4)$$

When $d = 1$, the solution of (4) is unique, and is given by

$$R_p(x) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \cosh^{-2/(p-1)}\left(\frac{p-1}{2}x\right). \quad (5)$$

When $d \geq 2$, Eq. (4) admits an infinite number of solutions. The solution with the minimal power, which we denote by $R^{(0)}$, is unique, and is called the ground state.

2.1. Critical NLS

In the critical case $(p-1)d = 4$, Eq. (1) can be rewritten as

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi = 0, \qquad \psi_0(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^1, \quad (6)$$

and Eq. (4) can be rewritten as

$$R''(r) + \frac{d-1}{r}R' - R + R^{4/d+1} = 0, \qquad R'(0) = 0, \qquad R(\infty) = 0. \quad (7)$$

Theorem 1 (*Weinstein [10]*). *A sufficient condition for global existence in the critical NLS (6) is $\|\psi_0\|_2^2 < P_{\text{cr}}$, where $P_{\text{cr}} = \|R^{(0)}\|_2^2$, and $R^{(0)}$ is the ground state of Eq. (7).*

The critical NLS (6) admits the explicit solution

$$\psi_{\text{explicit}}(t,r) = \frac{1}{L^{d/2}(t)} R^{(0)}\left(\frac{r}{L(t)}\right) e^{i\tau + i\frac{L_t}{L}\frac{r^2}{4}}, \quad (8a)$$

where

$$L(t) = T_c - t, \qquad \tau(t) = \int_0^t \frac{1}{L^2(s)}\, ds = \frac{1}{T_c - t}. \quad (8b)$$

More generally, applying the dilation transformation with $\lambda = \alpha$ and the temporal translation $T_c \longrightarrow \alpha^2 T_c$ shows that the critical NLS (6) admits the explicit solutions

$$\psi_{\text{explicit},\alpha}(t,r) = \frac{1}{L_\alpha^{d/2}(t)} R^{(0)}\left(\frac{r}{L_\alpha(t)}\right) e^{i\tau_\alpha + i\frac{(L_\alpha)_t}{L_\alpha}\frac{r^2}{4}}, \quad (9a)$$

where

$$L_\alpha(t) = \alpha(T_c - t), \qquad \tau_\alpha(t) = \int_0^t \frac{1}{L_\alpha^2(s)}\, ds = \frac{1}{\alpha^2}\frac{1}{T_c - t}, \quad (9b)$$

$\alpha > 0$.

The explicit solutions (8)–(9) become singular at $t = T_c$. These solutions are unstable, however, as the have exactly the critical power for collapse. Therefore, any infinitesimal perturbation which decreases their power, will arrest the collapse.

[1] We call the L^2 norm the *power*, since in optics it corresponds to the beam's power.