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Saddle–node bifurcation of invariant cones in 3D piecewise linear systems☆

Victoriano Carmona *, Soledad Fernández-García, Emilio Freire

Departamento de Matemática Aplicada II, Universidad de Sevilla, Escuela Técnica Superior de Ingeniería, Camino de los Descubrimientos s/n, 41092 Seville, Spain

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ABSTRACT

In this work, the existence of a saddle–node bifurcation of invariant cones in three-dimensional continuous homogeneous piecewise linear systems is considered. First, we prove that invariant cones for this class of systems correspond one-to-one to periodic orbits of a continuous piecewise cubic system defined on the unit sphere. Second, let us give the conditions for which the sphere is foliated by a continuum of periodic orbits. The principal idea is looking for the periodic orbits of the continuum that persist when this situation is perturbed. To do this, we establish the relationship between the invariant cones of the three-dimensional system and the periodic orbits of two planar hybrid piecewise linear systems. Next, we define two functions whose zeros provide the invariant cones that persist under the perturbation. These functions will be called Melnikov functions and their properties allow us to state some results about the existence of invariant cones and other results about the existence of saddle–node bifurcations of invariant cones, which is the principal goal of this paper.

1. Introduction and preliminary results

Nowadays, piecewise linear systems are being extensively studied because they are able to model several mechanical and electronic elements and even some biological behaviors [1–6]. Moreover, piecewise linear systems seem to be able to reproduce the dynamical behavior of a nonlinear general system [7,8,5]. On the other hand, piecewise linear systems can be used as a tool to understand some basic bifurcations that have their starting point in the change of stability of one equilibrium point [9–13]. In some of the previous works, it is shown that the change of the stability of one equilibrium point forces the appearance of a limit cycle. Indeed, to analyze this phenomenon it is necessary to study, in some situation, the behavior of the equilibria on the separation boundaries. The topological type of these equilibria is essential to ensure (or not) the existence of limit cycles. When the system is planar continuous and piecewise linear with two zones of linearity, this behavior is well known [10,11]. However, the problem is compounded for continuous three-dimensional systems. For instance, in [14] the authors prove that the continuous matching of two stable linear systems can be unstable. The instability of the origin, the unique equilibrium point of the system, can occur when the system has an invariant cone; by contrast, the absence of invariant cones ensure the stability of the origin when the matrices of the system are Hurwitzian [15,14]. This is a generalization of Theorem 3.4 of [16]. Therefore, it is important to study the existence of the invariant cones for this class of systems. We realize that these invariant manifolds can be considered as the center manifold, in non-generic cases, for non-smooth systems and so, the invariant cones have to play an important role. As a remark related, the stability of the origin can be guaranteed, as it is well known, by means of Lyapunov functions. However, it is not easy to find Lyapunov functions for piecewise linear systems, even when the involved matrices are Hurwitzian [17,18].

In [15], the authors provide a rigorous analysis about the existence of invariant cones for continuous homogeneous three-dimensional piecewise linear systems with two linearity zones separated by a plane. In that work, the invariant cones are classified as two-zonal ones when they live in both zones of linearity and one-zonal ones in the other case. As it is established in [15], the one-zonal invariant cones are not isolated and the maximum number of two-zonal invariant cones is two. Furthermore, the authors conjecture the existence of a saddle–node bifurcation of two-zonal invariant cones for an adequate choice of the parameters of the system. The principal aim of this work is to prove this conjecture, giving the local expressions of the hyper-surface (in the parameter space), where the saddle–node bifurcation occurs.

To start, we work with a normal form for piecewise linear systems. Concretely, we want a piecewise linear system which

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* Corresponding author. Tel.: +34 954555913; fax: +34 954486165.
E-mail addresses: vcarmona@us.es (V. Carmona), soledad@us.es (S. Fernández-García), efrem@us.es (E. Freire).

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cannot be decoupled and thus it cannot be reduced to a lower dimension problem. These systems are called proper from pioneer works of Chua and collaborators (see, for example, [6]). As it is proven in [2], any proper three-dimensional continuous homogeneous piecewise linear system with two linearity zones separated by a plane can be written into the Lienard form

$$\dot{\mathbf{x}} = F(\mathbf{x}) = \begin{cases} M^+\mathbf{x} & \text{if } x \geqslant 0, \\ M^-\mathbf{x} & \text{if } x < 0, \end{cases} \tag{1}$$

where $\mathbf{x} = (x, y, z)^T$,

$$M^+ = \begin{pmatrix} t^+ & -1 & 0 \\ m^+ & 0 & -1 \\ d^+ & 0 & 0 \end{pmatrix}, \qquad M^- = \begin{pmatrix} t^- & -1 & 0 \\ m^- & 0 & -1 \\ d^- & 0 & 0 \end{pmatrix},$$

being $t^\pm, m^\pm$ and $d^\pm$ the coefficients of the characteristic polynomials of matrices $M^\pm$.

To analyze the dynamical behavior of piecewise linear systems, it is usual to introduce an adequate Poincaré map defined in the separation plane $\Pi \equiv \{x = 0\}$ by using some Poincaré half-maps. A detailed study of Poincaré half-maps can be found in [19–21]. Now, a Poincaré map for system (1) will be defined.

For every point $\mathbf{p} = (x_\mathbf{p}, y_\mathbf{p}, z_\mathbf{p})^T \in \mathbb{R}^3$ we denote by $\mathbf{x_p}(t) = (x_\mathbf{p}(t), y_\mathbf{p}(t), z_\mathbf{p}(t))^T$ the solution of system (1) with initial condition $\mathbf{x_p}(0) = \mathbf{p}$. The corresponding orbit is denoted by $\gamma_\mathbf{p}$.

If $x_\mathbf{p} = 0$ and $e_1^T M^+\mathbf{p} = y_\mathbf{p} > 0$, where $e_1 = (1, 0, 0)^T$, then the orbit $\gamma_\mathbf{p}$ crosses transversally the plane Π with $x_\mathbf{p}(-t) < 0$ and $x_\mathbf{p}(t) > 0$ for $t > 0$ small enough. If $x_\mathbf{p}(t)$ vanishes in $(0, +\infty)$, then we define the right flying half-time $\tau_\mathbf{p}^+$ as the positive value such that $x_\mathbf{p}(\tau_\mathbf{p}^+) = 0$ and $x_\mathbf{p}(t) > 0$ in $\left(0, \tau_\mathbf{p}^+\right)$. In such a case, we define the right Poincaré half-map $\mathcal{P}^+$ at the point $(y_\mathbf{p}, z_\mathbf{p})$ as $\mathcal{P}^+(y_\mathbf{p}, z_\mathbf{p}) = \left(y_\mathbf{p}(\tau_\mathbf{p}^+), z_\mathbf{p}(\tau_\mathbf{p}^+)\right)^T$. Note that the right Poincaré half-map $\mathcal{P}^+$ depends only on the linear system $\dot{\mathbf{x}} = M^+\mathbf{x}$.

Analogously, we can define the left flying half-time $\tau_\mathbf{p}^-$ and the left Poincaré half-map $\mathcal{P}^-$ when $x_\mathbf{p} = 0$ and $e_1^T M^-\mathbf{p} = y_\mathbf{p} < 0$. Therefore, one can introduce the Poincaré map as $\mathcal{P} = \mathcal{P}^+ \circ \mathcal{P}^-$ whose domain is contained in $\mathcal{D} = \{\mathbf{p} \in \Pi : e_1^T M^-\mathbf{p} < 0, e_1^T M^+\mathcal{P}^-(\mathbf{p}) > 0\}$.

Taking into consideration that the vector field F of system (1) is homogeneous, i.e. $F(\mu\mathbf{x}) = \mu F(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$ and $\mu \geqslant 0$, it is easy to see that the maps $\mathcal{P}^+$, $\mathcal{P}^-$ and $\mathcal{P}$ are also homogeneous and hence, these maps transform half-straight lines contained in the plane $x = 0$ and passing through the origin into half-straight lines contained in the plane $x = 0$ that also pass through the origin (see Fig. 1). Thus, if the Poincaré map $\mathcal{P}$ possesses an invariant half-straight line we say that the system (1) has a two-zonal invariant cone. In Fig. 2 it is shown a two-zonal invariant cone and a continuum of one-zonal invariant cones. In the last case, the system has an invariant cone intersecting tangentially the separation plane.

Furthermore, it is possible to define a map $\mathcal{S}^-$ that transforms the slopes of the initial half-straight lines into the slopes of the final half-straight lines by means of $\mathcal{P}^-$. Similarly, we can also define a map $\mathcal{S}^+$ by considering the slopes of initial and final half-straight lines by applying the right Poincaré half-map $\mathcal{P}^+$. Hence, system (1) has an invariant two-zonal cone if and only if the map $\mathcal{S} = \mathcal{S}^+ \circ \mathcal{S}^-$ has a fixed point or equivalently, the generalized eigenvalue problem $\mathcal{P}(\mathbf{v}) = \delta\mathbf{v}$ has solution for $\delta > 0$ with $(0, \mathbf{v}) \in \mathcal{D}$. An analysis of maps $\mathcal{S}^+$ and $\mathcal{S}^-$ can be found in [15], where the authors provide a parametric representation of these maps as functions of the flying half-times. In [22], the author studies the eigenvalue problem to give some bifurcations of periodic orbits which live in the invariant cones; in particular, one generalization of the Hopf bifurcation is analyzed.

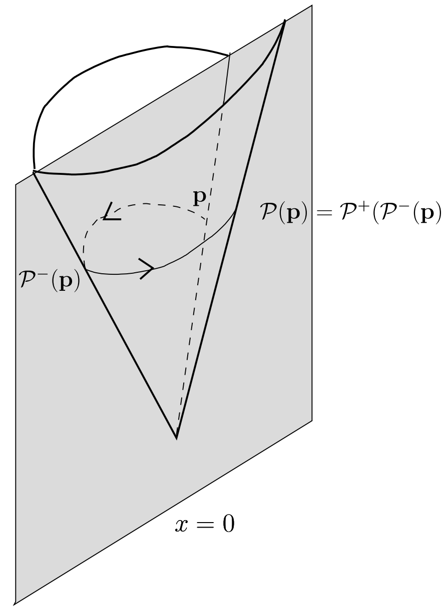

Fig. 1. Poincaré half-maps $\mathcal{P}^+$ and $\mathcal{P}^-$ and Poincaré map $\mathcal{P}$ of system (1).

On the other hand, if the flow of system (1) is projected onto the unit sphere S^2, then the invariant cones of the system can be considered as periodic orbits of a suitable system defined on the unit sphere. To see this, by following the original work of Hadeler [23], it is just necessary to do the change of variables $\mathbf{u} = \mathbf{x}\|\mathbf{x}\|^{-1}$, $r = \|\mathbf{x}\|$, for $\mathbf{x} \neq \mathbf{0}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. Then, we obtain the system

$$\begin{cases} \dot{\mathbf{u}} = F(\mathbf{u}) - \langle F(\mathbf{u}), \mathbf{u}\rangle\mathbf{u}, & \mathbf{u} \in S^2, \\ \dot{r} = \langle F(\mathbf{u}), \mathbf{u}\rangle\, r, & r \geqslant 0 \end{cases}$$

being $\langle\cdot,\cdot\rangle$ the ordinary scalar product in $\mathbb{R}^3$. Now, it is immediate to observe that the invariant cones of system (1) correspond one-to-one to the periodic orbits of the following continuous piecewise cubic system on S^2

$$\dot{\mathbf{u}} = F(\mathbf{u}) - \langle F(\mathbf{u}), \mathbf{u}\rangle\mathbf{u}, \quad \mathbf{u} \in S^2. \tag{2}$$

Here, it is also possible to define the periodic orbits as one-zonal and two-zonal ones. By using, for example, system (2), one can prove that the one-zonal invariant cones of (1) cannot be isolated and in this case matrix M^+ (or M^-) has complex eigenvalues with the real part of the complex eigenvalues and the real eigenvalue shared. Moreover, when system (1) possesses one invariant cone living in each zone of linearity, then the sphere is foliated by a continuum of periodic orbits when the traces of matrices M^+ and M^- coincide. Here, two invariant cones tangent to the separation plane appear. These statements are stated in the next results and are deduced from Proposition 6 and statement (b) of Theorem 2 in [15].

Proposition 1. *Assume that the eigenvalues of the matrices of system (1), M^+ and M^-, are $\lambda^-, \alpha^- \pm i\beta^-$ and $\lambda^+, \alpha^+ \pm i\beta^+$, respectively, with $\lambda^-, \alpha^-, \beta^-, \lambda^+, \alpha^+, \beta^+ \in \mathbb{R}, \beta^- > 0$ and $\beta^+ > 0$. Then, the following statements hold.*

(a) *If system (1) has an one-zonal invariant cone $\mathcal{C}$ living in the half-space $\{x \leqslant 0\}$, then $\alpha^- = \lambda^-$ and the system has a continuum of one-zonal invariant cones living in the zone $\{x \leqslant 0\}$.*
(b) *If system (1) has an one-zonal invariant cone $\mathcal{C}$ living in the half-space $\{x \geqslant 0\}$, then $\alpha^+ = \lambda^+$ and the system has a continuum of one-zonal invariant cones living in the zone $\{x \geqslant 0\}$.*