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Dynamics of poles with position-dependent strengths and its optical analogues

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ABSTRACT

The dynamics of point vortices is generalized in two ways: first by making the strengths complex, which allows for sources and sinks in superposition with the usual vortices, second by making them functions of position. These generalizations lead to a rich dynamical system, which is nonlinear and yet has conservation laws coming from a Hamiltonian-like formalism. We then discover that in this system the motion of a pair mimics the behavior of rays in geometric optics. We describe several exact solutions with optical analogues, notably Snell's law and the law of reflection off a mirror, and perform numerical experiments illustrating some striking behavior.

Introduction

The dynamics of point vortices with fixed strengths in a 2-dimensional ideal fluid has a classical pedigree, e.g. [1], Art. 154–160. The subject continues to be actively pursued: a modern survey [2] on its equilibrium aspect alone lists more than 100 papers.

We generalize vortex dynamics in two ways, firstly allowing, besides vortices, sources/sinks as well as their superpositions ('poles'), and secondly allowing the strengths of these poles to vary as functions of position in the plane.

The first generalization goes back to a 1928 paper by Friedman and Polubarinova–Kochina (the former is the same Friedman as in the eponymous cosmological model). The rather more recent paper by Borisov and Mamaev [3] contains references as well as a good theoretical analysis. There is also a very readable account of the history, derivations and generalizations of point vortex models in the paper by Llewellyn Smith in this issue [4]. Lacomba [5] has also recently studied interactions between vortices and sources/sinks. Here we present a couple of new exact solutions and alternative derivations of some old ones.

The second generalization seems less explored, and leads to rich dynamics, which we illustrate with a variety of exact solutions with *analogues in geometric optics*, the position-dependent strength of a pole replacing the medium-dependent index of refraction. As typical examples we detail the analogues of Snell's law and the law of reflection off a mirror, in generalized forms. Optical analogy is not so obvious: though it was suggested by Kimura [6] that in the dipole limit a classical vortex pair should travel along a geodesic, the principle governing light rays in geometric optics is one of least time, not of least length. Nevertheless, the dynamics of poles with position-dependent strengths turns out to be quite versatile in mimicking the ray representation of phenomena of wave propagation. Take for instance the work by Berry [7] on focusing and defocusing of surface waves by underwater landscape. It will become clear that such effects are realizable by our dynamics, too. There are also similarities with results of Longuet-Higgins on trapping waves around islands [8].

In an interesting paper, Hinds et al. [9] consider the dynamics of a pair of vortex patches as they cross a step change in the depth of the fluid, and also find that the pair is refracted provided they are sufficiently well separated compared to their size and the angle of incidence, otherwise they find vortex shedding. While we do not claim that our introduction of a 'seabed' function genuinely models a variation in depth, we do consider only point vortices (and poles) so vortex shedding would not arise.

In the language of dynamical systems, this second generalization through the introduction of a 'seabed' function S in Section 4 puts us in the realm of *hybrid systems*, where different equations of motion govern different regions of the phase space.

It may not be amiss to point out that the dynamical system (2), which is the chief object of our study, is quite nonlinear—in a sense more so than say the Euler or Navier–Stokes equations. In the latter, indeed, the nonlinearity is separated out as an additive term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\left(\frac{1}{2}\boldsymbol{v}^2\right) - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})$, so that we can resort to linearization by dropping this nonlinear term or by substituting for it a term (background flow $\cdot\,\nabla)\boldsymbol{v}$ (Stokes and Oseen approximations, respectively). In contrast, our system is so nonlinear that it is not even clear whether or not there exists a 'linearization' of the system that makes sense.

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In the light of this inseparable nonlinearity, it is noteworthy that our system proves to be analysable in elementary terms, and many instances of complete integrability explicitly spelled out. This is thanks, ultimately, to the fact that the classical vortex dynamics is *Hamiltonian* and our generalization is *something like Hamiltonian*.

1. Equation of motion

We begin with a discussion of 2-dimensional ideal fluid flow in terms of complex potential, but shall specialize soon. Consider N interacting points $z_1, \ldots, z_N \in \mathbb{C}$ called *poles*, each pole z_i carrying with it a family of complex-valued functions $\{\mu_n^i(z)\}_{n\in\mathbb{Z}}$ called *strengths*, only finitely many of which are nonzero. The poles move according to

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \sum_{j:j\neq i} \sum_{n\in\mathbb{Z}} \overline{\mu_n^j\big(z_j(t)\big)\big(z_i(t)-z_j(t)\big)^n} \qquad (1)$$

for $i = 1, \ldots, N$ ($\bar{}$ denotes complex conjugation). The dynamics of (1), being 1st-order in time t, has *no inertia*, in the sense that the poles' instantaneous positions determine their velocities: the phase space is a product of N copies of $\mathbb{C}$ (minus diagonals if we wish a priori to exclude collisions), not a (co)tangent bundle. We can set up a dynamical system like this on any domain of any Riemann surface. In simple domains that arise in practice, solutions can be found by the method of images.

A term of the form $\overline{\mu(z-z_j)^n}$ on the right side of (1) represents a flow velocity induced by z_j at z. The pictures for $n = -1$ have rotational symmetry: source or sink of flux $2\pi\mu$ for μ real, vortex of circulation $2\pi\sqrt{-1}\mu$ for μ pure imaginary, in general a superposition of these, i.e. a spiral node. Other n exhibit other symmetries: multipolar flows for $n < -1$, uniform flow for $n = 0$, and corner flows for $n > 0$.

2. Homogeneous systems, conservation laws

Now suppose (1) is homogeneous so that $\mu_n^i = 0$ except for a certain exponent $n = n_0$, and moreover all $\mu^i := \mu_{n_0}^i$ are fixed. Then (1) may be recast in the 'canonical' form

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i = \frac{2}{\overline{\mu^i}} \frac{\partial}{\partial \overline{z_i}} H,$$

with

$$H(z_1, \ldots, z_N) = \mathrm{Re} \sum_{i,j:i<j} \mu^i \mu^j G(z_i - z_j)$$

and

$$G(z_i - z_j) = \begin{cases} \dfrac{1}{n_0+1}(z_i - z_j)^{n_0+1} & \text{when } n_0 \neq -1, \\ \log(z_i - z_j) & \text{when } n_0 = -1 \end{cases}$$

(G as in 'Green'). From

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H\big(z_1(t), \ldots, z_N(t)\big) &= \sum_i \left(\frac{\partial}{\partial z_i} H \cdot \frac{\mathrm{d}}{\mathrm{d}t} z_i + \frac{\partial}{\partial \overline{z_i}} H \cdot \frac{\mathrm{d}}{\mathrm{d}t} \overline{z_i}\right) \\ &= \sum_i \mathrm{Re}\,(\mu) \left|\frac{\mathrm{d}}{\mathrm{d}t} z_i(t)\right|^2\end{aligned}$$

we see

Theorem 2.1. *If all μ^i are pure imaginary and fixed, then H is conserved.*

Next let the homogeneity degree n_0 be odd, with μ^i still fixed. Pairwise cancelation in (1) yields $\sum_i \overline{\mu^i} \mathrm{d}z_i/\mathrm{d}t = 0$, whence

Theorem 2.2. *If the degree of homogeneity is odd and $\mu := \sum_i \mu^i \neq 0$, then the 'center of strength'*

$$c = \frac{\sum_i \overline{\mu^i} z_i}{\overline{\mu}}$$

is conserved. If $\mu = 0$, then for every partition of the index set $I \sqcup I' = \{1, \ldots, N\}$ such that $\mu_I := \sum_{i\in I} \mu^i \neq 0$ and $\mu_{I'} := \sum_{i'\in I'} \mu^{i'} \neq 0$, the difference between the 'subcenters'

$$\frac{\sum_{i\in I} \overline{\mu^i} z_i}{\overline{\mu_I}} - \frac{\sum_{i'\in I'} \overline{\mu^{i'}} z_{i'}}{\overline{\mu_{I'}}}$$

is conserved.

The 'partition' part of this Theorem is elementary but does not appear to have been used prior to the paper by Montaldi et al. [10].

If instead the homogeneity degree n_0 is even and there are just 2 poles z, z' with strengths μ, μ', then $\overline{\mu} z - \overline{\mu'} z'$ is a conserved quantity. However, this does not appear to extend to more than 2 poles.

Finally, how can we extend the affine symmetry of the phase space to that of the phase space–time so as to preserve the invariance of (1)? The requirement that time t be real gives the answer.

Theorem 2.3. *The system (1) is invariant under the action of $\mathbb{C}^* \ltimes \mathbb{C}$ if and only if it is homogeneous of degree $n_0 = -1$: here $(a, b) \in \mathbb{C}^* \ltimes \mathbb{C}$ acts by sending (t, z) to $(|a|^2 t, az + b)$.*

3. Exact solutions

We shall, in the remainder of the paper, focus on the theory where (1) involves only the exponent $n_0 = -1$. In this section we suppose all the strengths are fixed: $\mu_{-1}^i(z) = \mu^i$. Thus the equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \sum_{j:j\neq i} \frac{\overline{\mu^j}}{\overline{z_i(t)} - \overline{z_j(t)}} \qquad (i = 1, \ldots, N). \qquad (2)$$

We consider position-dependent strengths in Sections 4 and 5. When all μ^i are pure imaginary, we are back to classical point vortices and recover the logarithmic H as their Hamiltonian.

3.1. Self-similar solutions

If a collection of poles happens to move in a self-similar manner, then Theorem 2.3 reduces the (complex) degree of freedom from N to 1, down to a single equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z = \frac{M}{\overline{Z}}, \qquad (3)$$

or in polar coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} |Z|^2 = \mathrm{Re}\, M, \qquad |Z|^2 \frac{\mathrm{d}}{\mathrm{d}t} \arg Z = \mathrm{Im}\, M.$$

The solution is

$$Z(t) = T \exp\left(\sqrt{-1} \frac{\mathrm{Im}\, M}{\mathrm{Re}\, M} \log T\right) Z(0), \qquad (4)$$

where

$$T = \sqrt{1 + \frac{\mathrm{Re}\, M}{|Z(0)|^2/2} t} \quad \text{if } \mathrm{Re}\, M \neq 0,$$