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# Kinematic relative velocity with respect to stationary observers in Schwarzschild spacetime

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ABSTRACT

We study the kinematic relative velocity of general test particles with respect to stationary observers (using spherical coordinates) in Schwarzschild spacetime, obtaining that its modulus does not depend on the observer, unlike Fermi, spectroscopic, and astrometric relative velocities. We study some fundamental particular cases, generalizing some results given in other work about stationary and radial free-falling test particles. Moreover, we give a new result about test particles with circular geodesic orbits: the modulus of their kinematic relative velocity with respect to any stationary observer depends only on the radius of the circular orbit, and so it remains constant.



## 1. Introduction

The concept of "relative velocity" of a test particle with respect to an observer in general relativity is only well defined when the observer and the test particle are in the same event. Nevertheless, the notion of relative velocity of a distant test particle is fundamental in physics and so it was revised by the IAU using reference systems adapted to the solar system (see [1,2]). Thereby, some authors have introduced new geometric concepts motivated by the coordinate-dependence of some definitions; for example, scaled Fermi–Walker derivatives let us define geometrically local notions of velocities of a test particle with respect to a congruence of observers (see [3]). Moreover, four different intrinsic geometric definitions of relative velocity of a distant test particle with respect to a single observer were introduced in [4]. These definitions are strongly associated with the concept of simultaneity: *kinematic* and *Fermi* in the framework of "spacelike simultaneity", *spectroscopic* and *astrometric* in the framework of "lightlike simultaneity". These four concepts each have full physical sense, and have proved to be useful in the study of properties of particular spacetimes (see [4–7]).

Following this line, we are going to study the kinematic relative velocity of test particles with respect to stationary observers in Schwarzschild spacetime. This velocity shows a kind of "Newtonian behavior" in this spacetime unlike the other three velocities, and some interesting properties about stationary observers hold, as we are going to develop in the present work.

This paper is organized as follows. In Section 2, we establish notation and define the concept of kinematic relative velocity. In Section 3 we introduce the Schwarzschild metric in spherical coordinates and their corresponding stationary

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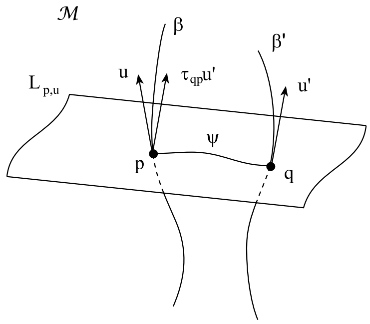


**Fig. 1.** Scheme of the elements involved in the definition of the kinematic relative velocity of $\beta'$ (test particle) with respect to $\beta$ (observer). The curve $\psi$ is a spacelike geodesic orthogonal to the 4-velocity of $\beta$ at $p$, denoted by $u$. The vector $u'$ is the 4-velocity of $\beta'$ at $q$.

observers, giving some lemmas that are applied in Section 4 to obtain the main result: the expression of the modulus of the kinematic relative velocity of a general test particle with respect to any stationary observer. We also study in this section some fundamental examples that were previously introduced in [4], generalizing the results obtained in that paper, and we present a new example about circular geodesic orbits. Finally, in Section 5 we give a property that extends the main result to general relativity.

## 2. Definitions and notation

We work in a Lorentzian spacetime manifold $(\mathcal{M}, g)$, with $c = 1$ and using the "mostly plus" signature convention $(-, +, +, +)$. We suppose that $\mathcal{M}$ is a convex normal neighborhood; thus, given two events $p$ and $q$ in $\mathcal{M}$, there exists a unique geodesic joining them (results on the existence of convex normal neighborhoods in semi-Riemannian manifolds are given in [8, pp. 129–131]; see Remark 2.1 for a discussion about working in a non-convex normal neighborhood). The parallel transport from $q$ to $p$ along this geodesic is denoted by $\tau_{qp}$. If $\beta : I \to \mathcal{M}$ is a curve with $I \subseteq \mathbb{R}$ a real interval, we identify $\beta$ with the image $\beta I$ (that is a subset in $\mathcal{M}$), in order to simplify the notation. Vector fields are denoted by uppercase letters and vectors (defined at a single point) are denoted by lowercase letters. Moreover, if $x$ is a spacelike vector, then $\|x\| := g\,(x, x)^{1/2}$ is the modulus of $x$. If $X$ is a vector field, $X_p$ denotes the unique vector of $X$ in $T_p\mathcal{M}$.

In general, we say that a timelike world line $\beta$ is an *observer* (or a *test particle*). Nevertheless, we say that a future-pointing timelike unit vector $u$ in $T_p\mathcal{M}$ is an *observer at p*, identifying the observer with its 4-velocity.

The *Landau submanifold* $L_{p,u}$ (also called *Fermi surface*) is given by all the geodesics starting from $p$ and orthogonal to $u$ (see [9,10,6]).

### 2.1. Kinematic relative velocity

Throughout the paper, we consider an observer $\beta$ and a test particle $\beta'$ (parameterized by their proper times) with 4-velocities $U$ and $U'$ respectively. Let $u := U_p$ be the 4-velocity of $\beta$ at an event $p$ and let $q$ be the event of $\beta'$ such that there exists a spacelike geodesic $\psi$ orthogonal to $u$ joining $p$ and $q$ (see Fig. 1). Note that since we work in a convex normal neighborhood, this event is unique and it is given by $q := L_{p,u} \cap \beta'$. We denote $u' := U'_q$ in order to simplify the notation.

The *kinematic relative velocity of* $u'$ *with respect to* $u$ is the vector

$$v_{\mathrm{kin}} := \frac{1}{-g\left(\tau_{qp}u', u\right)}\tau_{qp}u' - u. \tag{1}$$

In the case $p = q$, this definition coincides with the usual concept of relative velocity

$$v = \frac{1}{-g\,(u', u)}u' - u, \tag{2}$$

which is only defined when $u$ and $u'$ are in the same tangent space.

Note that $v_{\mathrm{kin}}$ is spacelike and orthogonal to $u$, and the square of its modulus is given by

$$\|v_{\mathrm{kin}}\|^2 = g\,(v_{\mathrm{kin}}, v_{\mathrm{kin}}) = 1 - \frac{1}{g\left(\tau_{qp}u', u\right)^2} = 1 - \frac{1}{g\left(u', \tau_{pq}u\right)^2}, \tag{3}$$

since parallel transport conserves the metric. Varying $p$ along $\beta$, we construct the vector field $V_{\mathrm{kin}}$ defined on $\beta$, representing the *kinematic relative velocity of* $\beta'$ *with respect to* $\beta$ (see [4,11]). Throughout the paper we are going to denote $v_{\mathrm{kin}} := V_{\mathrm{kin}\,p}$ as we have already done in this section.