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# Classical and quantum Bianchi type III vacuum Hořava–Lifshitz cosmology

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ABSTRACT

A diagonal Bianchi type III spacetime is treated, both at the classical and quantum level, in the context of *Hořava–Lifshitz* gravity. The system of the classical equations of motion is reduced to one independent Abel's equation of the first kind. Closed form solutions are presented for various values of the coupling constants appearing in the action. Due to the method used, solutions of Euclidean, Lorentzian and neutral signatures are attained. The main general solution for $\lambda = 1$ is seen to contain a curvature singularity at $t = 0$, while in its Einsteinian limit it reproduces the known diagonal solution (with and/or without cosmological constant). The rest of the solutions correspond to somewhat isolated cases, as they emerge from either setting the coefficient of the lapse to zero or taking $\lambda = 1/3$. They are singularity free, but they all have a singular limit as the other constants approach their Einsteinian values. At the quantum level, the resulting Wheeler–DeWitt equation is explicitly solved for $\lambda = 1$, $\sigma = 0$ and $\lambda = \frac{1}{3}$. The ensuing wave functions diverge in the Einsteinian limit.



## 1. Introduction

It could be stated that an important feature of general relativity is the covariance of the theory under four dimensional diffeomorphisms of the spacetime manifold. Recently, a new theory of gravity was proposed, with the aim of being complete in the UV [1]. Its basic assumption is the admittance that the aforementioned covariance is not a fundamental property of the theory, but arises rather accidentally in the context of lower energies. The resulting geometry is then exhibiting a non-isotropic time and space scaling invariance

$$t \mapsto b^z t, \qquad x^i \mapsto b x^i.$$

As a consequence, one must start from an action functional and equations of motion which involve higher derivatives in the spatial coordinates. On the other hand, it is an advantage that there are no higher derivatives in time. A theory was constructed for various values of the critical exponent $z$ ($z = 2$ [2], $z = 3$ and $z = 4$ [1]).

In the context of this theory one has to consider a four dimensional differentiable manifold, $\mathcal{M}$, with a codimension one foliation $\mathcal{F}$ and admit as the "gauge invariance" group the foliation preserving diffeomorphisms, i.e. coordinate transformations of the restricted form

$$\tilde{t} = f(t), \qquad \tilde{x}^i = g^i(t, x^j). \tag{1.1}$$

With this reasoning the author of [1] modified the 3+1 decomposed Einstein–Hilbert action by adding one extra coupling constant in the kinetic term that destroys four dimensional covariance. As far as the potential part is concerned, extra terms

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were added under the condition of detailed balance. The latter results in a potential term of the Lagrangian density, which up to a coupling constant is

$$\mathcal{L}_{\text{potential}} = E_{ij} G^{ijkl} E_{kl}$$

with $G^{ijkl}$ the generalized Wheeler–DeWitt supermetric and $E_{ij}$ a three-tensor that is obtained by a variational principle $\sqrt{g} E^{ij} = \frac{\delta W(g_{kl})}{\delta g_{ij}}$ for some action $W$ [1].

There are many papers studying certain cosmological implications of the theory, mostly for a FRW metric, either free of matter or with a scalar field (see for example [3–9], or, for an excellent review on Hořava–Lifshitz cosmology, [10] and the references therein). The Bianchi types are in general anisotropic, the anisotropies considered to be present in early stages of the universe evolution. A systematic way to obtain the classical solutions in the Einstein case has been initiated in [11] where the automorphisms of the Lie algebra of the symmetry group of each Bianchi type have been revealed as kinematically induced by particular spacetime coordinate transformations. The use of these automorphisms as Lie-point symmetries of the corresponding Einstein equations has been put forward for types I, II, III, IV, V and VII in [12–15]. This application resulted in the discovery of the full solution space expressed in the form of a Painlevé VI transcendental. In this paper we intend to use the same method for obtaining classical solutions of the Hořava–Lifshitz cosmology, with the motivation of examining if the nonsingular character of earlier found solutions persists in the presence of geometrical anisotropy. It is thus clear that we are looking for a Bianchi type which is anisotropic, more "difficult" than I, II and V and at the same time is manageable from the point of view of analytically solving it. The type III certainly belongs to the difficult cases and its diagonal (LRS) specialization is analytically treatable. Therefore, in the present work we investigate the solution space for a vacuum diagonal Bianchi type III model in the context of Hořava–Lifshitz cosmology and compare with analogous solutions to Einstein's equations when the limit exists. Moreover, we proceed with the canonical quantization of the axisymmetric case and derive the Wheeler–DeWitt equation for this cosmological model, giving a solution for specific values of some of the coupling constants. The results obtained are also compared to those obtained by quantizing the Einsteinian action.

## 2. Equations of motion

Our starting point is the action [3]

$$S = \int dt d^3x \sqrt{g} N \left( \alpha (K_{ij}K^{ij} - \lambda K^2) + \frac{\beta}{g} C_{pq}C^{pq} + \gamma \frac{\epsilon^{ipk}}{\sqrt{g}} R_{il} R^l{}_{k;p} + \zeta R^i{}_j R^j{}_i + \eta R^2 + \xi R + \sigma \right) \tag{2.1}$$

where $g$ is the determinant of the three dimensional metric $g_{ij}$, $N$ is the lapse function, $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - N_{i;j} - N_{j;i})$ the extrinsic curvature corresponding to the spatial metric, $K$ its trace, $C^{pq} = \epsilon^{pkl}(R^q_l - \frac{1}{4}R\delta^q{}_l)_{;k}$ the Cotton–York tensor density and $R_{ij}$, $R$ the Ricci tensor and scalar respectively. Each term in (2.1) comes with its own coupling constant $\alpha$, $\lambda$, $\beta$, $\gamma$, $\zeta$, $\eta$, $\xi$ and $\sigma$. The two parts of the kinetic term are distinguished by the existence of $\lambda$. For $\lambda = 1$ we get the kinetic term of Einstein's theory and the difference is restricted to the extra terms in the potential part and the different coupling constants that bind them together. We note here that ";" stands for covariant differentiation with respect to the spatial metric and all the latin indices run from 1 to 3.

Variation of the action by $\delta g_{mn}$ results in the following spatial equations of motion

$$\begin{aligned}
E^{mn} \equiv & -\frac{\beta N}{2g} g^{mn} C_{pq} C^{pq} - \beta \frac{\epsilon^{qkl}}{2g} (N C_{ql})_{;k} R^{mn} + \beta \frac{\epsilon^{qkl}}{4g} G^{rsmn} (N C_{ql})_{;ksr} \\
& - \frac{\beta}{2g} \left( N\epsilon^{qkm} C_q{}^l g^{sn} + N\epsilon^{qkl} C_q{}^m g^{sn} - N\epsilon^{qkn} C_q{}^m g^{sl} - N\epsilon^{qks} C_q{}^l g^{mn} + (m \leftrightarrow n) \right)_{;ksl} \\
& - \frac{\beta N}{4g} (\epsilon^{qkm} C_q{}^n R_{;k} + \epsilon^{qkn} C_q{}^m R_{;k}) \\
& + \frac{\beta}{2g} \left( N\epsilon^{qkl} C_q{}^m R^n_l + N\epsilon^{qml} C_q{}^k R^n_l - N\epsilon^{qnl} C_q{}^m R^k_l + (m \leftrightarrow n) \right)_{;k} \\
& - \gamma \frac{\epsilon^{ipk}}{2\sqrt{g}} N (R_i{}^m R^n{}_{k;p} + R_i{}^n R^m{}_{k;p}) + \frac{\gamma}{4\sqrt{g}} (N R_{qk;p})_{;sl} \Big( \epsilon^{mpk} g^{lq} g^{sn} + \epsilon^{lpk} g^{mq} g^{sn} \\
& - \epsilon^{lpk} g^{sq} g^{mn} - \epsilon^{mpk} g^{nq} g^{sl} + (m \leftrightarrow n) \Big) \\
& - \frac{\gamma}{4\sqrt{g}} \left( N\epsilon^{ipl} g^{sn} R_i{}^m + N\epsilon^{ipm} g^{sn} R_i{}^l - N\epsilon^{ips} g^{mn} R_i{}^l - N\epsilon^{ipn} g^{sl} R_i{}^m + (m \leftrightarrow n) \right)_{;psl}
\end{aligned}$$