Physica D 240 (2011) 1426–1448

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

On algebraic construction of certain integrable and super-integrable systems

A.J. Maciejewski [a,*], M. Przybylska [b], A.V. Tsiganov [c]

[a] J. Kepler Institute of Astronomy, University of Zielona Góra, Licealna 9, PL-65-417 Zielona Góra, Poland
[b] Institute of Physics, University of Zielona Góra, Licealna 9, PL-65-417 Zielona Góra, Poland
[c] St. Petersburg State University, St. Petersburg, Russia

ARTICLE INFO

Article history:
Received 25 November 2010
Accepted 29 May 2011
Available online 16 June 2011
Communicated by G. Stepan

Keywords:
Super-integrability
Bi-Hamiltonian systems
Poisson manifold
Hypergeometric function
Separation of variables

ABSTRACT

We propose a new construction of two-dimensional natural bi-Hamiltonian systems associated with a very simple Lie algebra. The presented construction allows us to distinguish three families of super-integrable monomial potentials for which one additional first integral is quadratic, and the second one can be of arbitrarily high degree with respect to the momenta. Many integrable systems with additional integrals of degree greater than two in momenta are given. Moreover, an example of a super-integrable system with first integrals of degree two, four and six in the momenta is found.

1. Introduction

The main aim of this paper is to study natural integrable systems

$$H_1 = 2p_1p_2 + V(q_1, q_2), \tag{1.1}$$

with two degrees of freedom associated with the various representations of Lie algebra

$$[N, a_+] = \kappa_1 a_+, \qquad [N, a_-] = -\kappa_2 a_-, \qquad [a_+, a_-] = 0 \tag{1.2}$$

labelled by parameters $\kappa_1, \kappa_2 \in \mathbb{R}$, and having the Casimir element

$$C = a_+^{\kappa_2} a_-^{\kappa_1}.$$

An obvious representation of this algebra is the algebra of smooth function defined on the phase space generated by

$$a_+ = q_1, \qquad a_- = q_2, \qquad N = -\kappa_1 q_1 p_1 + \kappa_2 q_2 p_2, \tag{1.3}$$

with bracket $[a, b] := \{a, b\}$, where $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket in $\mathbb{R}^4$.

We met this algebra investigating bi-Hamiltonian structures of two-dimensional natural Hamiltonian systems with homogeneous potentials and Newton's equations with homogeneous velocity independent forces. Namely, let us consider a natural system given by the Hamilton function of the form (1.1) with a monomial potential. More precisely, let us consider system governed by Hamiltonian of the form

$$H \equiv a_+ = 2p_1p_2 + q_1^a q_2^b, \tag{1.4}$$

where a and b are real parameters. For such a system the function

$$N = \alpha p_1 q_1 + \beta p_2 q_2, \tag{1.5}$$

satisfies the following equality

$$\{N, a_+\} = (\alpha + \beta)\, a_+,$$

* Corresponding author.
E-mail addresses: maciejka@astro.ia.uz.zgora.pl (A.J. Maciejewski), M.Przybylska@proton.if.uz.pl (M. Przybylska), andrey.tsiganov@gmail.com (A.V. Tsiganov).

doi:10.1016/j.physd.2011.05.020

provided that α and β fulfil equation

$$a\,\alpha + b\,\beta + \alpha + \beta = 0.$$

If we can find the remaining generator $a_- = H_2$, then we have integrable in the Liouville sense a dynamical system with homogeneous potential. In order to get this additional first integral we use the machinery of bi-Hamiltonian geometry, see [1,2].

It is worth noticing that the existence of function N with the prescribed property is related with a certain symmetry of the system. Let X_F denote the Hamiltonian vector field generated by F. Assume that for Hamiltonian (1.4) there exists function N such that $\{N, H\} = cH$, where c is a constant. Then the Hamiltonian vector field X_N is a master symmetry of X_H, as $[X_N, X_H] = cX_H$. Sometimes a master symmetry is called a conformal symmetry, see [3].

Let us assume that X_N with N of the following form

$$N := A(q_1, q_2)p_1 + B(q_1, q_2)p_2, \tag{1.6}$$

where $A(q_1, q_2)$ and $B(q_1, q_2)$ are differentiable functions, is a conformal symmetry of X_H. Then it is easy to show that necessarily $A = \alpha q_1$ and $B = \beta q_2$. Thus, equality $\{N, H\} = cH$ implies that

$$c = \alpha + \beta, \qquad \alpha a + \beta b + c = 0. \tag{1.7}$$

So, we recovered the assumed form of N, see (1.5). Now, the last relation in (1.2) tells that we need an additional integral $H_2 \equiv a_-$. Moreover, we require that $\{N, H_2\} = dH_2$, for a certain $d \in \mathbb{R}$, and it is equivalent that X_N is also a conformal symmetry of X_{H_2}.

We are going to work with systems (1.4) with two degrees of freedom because such systems appeared as subsystems on invariant manifold of n-dimensional Hamiltonian systems [4,5].

The plan of this paper is as follows. In the next section we start with a short description of how Hamiltonian systems of the form (1.4) appeared in our investigations of the integrability of natural Hamiltonian systems with homogeneous potentials. In Section 3 bi-Hamiltonian irregular Poisson manifolds as well as their application for construction of first integrals of considered systems are presented. The remaining sections contain results of the integrability analysis. In Section 4 four families of integrable systems with additional first integrals quadratic in the momenta are given. In the next four sections super-integrable cases in these families are distinguished. In Section 9 examples of integrable systems with additional first integrals of degree greater than two in the momenta are given. In Appendix we collected basic facts concerning the Gauss hypergeometric equation which are used in this paper.

2. Monomial potentials

Let us consider Hamiltonian systems with n degrees of freedom with the Hamiltonian of the following classical form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \tag{2.1}$$

where $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ are the canonical coordinates and momenta; the potential $V(\boldsymbol{q})$ is a homogeneous function of degree $k \in \mathbb{Q}$. The strongest necessary integrability conditions for such systems with $k \in \mathbb{Z}$ were obtained thanks to an application of differential Galois methods, see [6–8]. To derive conditions of this type one needs to know a particular non-equilibrium solution of the considered systems. For the systems given by Hamiltonian (2.1) a particular solution can be found in systematic way. Namely, if a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ satisfies

$$V'(\boldsymbol{d}) = \gamma \boldsymbol{d}, \tag{2.2}$$

for some $\gamma \in \mathbb{C}$, then the system has a particular solution of the following form

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \qquad \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d}, \tag{2.3}$$

where $\varphi(t)$ is a scalar function which is defined in the following way. If vector $\boldsymbol{d}$ satisfies Eq. (2.2) with $\gamma = 0$, then $\varphi(t) := t$, and $\boldsymbol{d}$ is called an improper Darboux point of potential V. In this case, if $k \in \mathbb{N}$, and the system is integrable in the Liouville sense, then all the eigenvalues of the Hessian matrix $V(\boldsymbol{d})$ vanish, see [5].

If the considered $\boldsymbol{d}$ satisfies Eq. (2.2) with $\gamma \neq 0$, then $\varphi(t)$ is a solution of equation $\ddot{\varphi} = -\varphi^{k-1}$. In this case $\boldsymbol{d}$ is called a proper Darboux point of V, and if the system is integrable, then for a given $k \in \mathbb{Z} \setminus \{-2, 2\}$, all eigenvalues of $V''(\boldsymbol{d})$ belong to a certain infinite subset of rational numbers $\mathbb{Q}$, see [6,7,4].

As we can see, in spite of the fact that the differential Galois theory is quite involved the final result has the form of simple arithmetic restrictions on the eigenvalues of matrix $V''(\boldsymbol{d})$. Moreover, for polynomial potentials some relations between eigenvalues of the Hessian calculated at all proper Darboux points exist. These relations together with arithmetic restrictions on the eigenvalues forced by necessary integrability conditions enable to find effectively explicit forms of integrable potentials at least for small n and k. Such a systematic analysis was initiated for $n = 2$ in [9,8] and later on it was developed for $n > 2$, see [5,10].

However, for some classes of potentials the above approach does not work. In particular it is a case if the considered potential does not have any proper Darboux point, and moreover, the Hessian matrix $V''(\boldsymbol{d})$ at each improper Darboux point is nilpotent. For $n = 2$ an almost complete characterisation of such polynomial potentials is given by the following lemma.

Lemma 2.1. *If a polynomial potential V of degree $k > 2$ does not have any proper Darboux point, then it is either equivalent to the following one*

$$V_{k,l} = \alpha(q_2 - \mathrm{i}q_1)^{k-l}(q_2 + \mathrm{i}q_1)^l, \quad \alpha \in \mathbb{C}^\star := \mathbb{C} \setminus \{0\} \tag{2.4}$$

for some $l = 2, \ldots, k - 2$, or $k = 2s$ and V has factor $(q_2 \pm \mathrm{i}q_1)$ with multiplicity s.

We say that potential V is equivalent to W iff $V(\boldsymbol{q}) = W(A\boldsymbol{q})$, where A is $n \times n$ matrix satisfying $AA^T = \beta \mathrm{Id}_n$ for a certain $\beta \in \mathbb{C}^\star$.