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On the geometric quantization of contact manifolds

S. Fitzpatrick *

Department of Mathematics, University of California, Berkeley, 749 Evans Hall #3840, Berkeley, CA 94720-3840, USA

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ABSTRACT

Suppose that (M, E) is a compact contact manifold, and that a compact Lie group G acts on M transverse to the contact distribution E. In an earlier paper, we defined a G-transversally elliptic Dirac operator $\not{D}_b$, constructed using a Hermitian metric h and connection ∇ on the symplectic vector bundle $E \to M$, whose equivariant index is well-defined as a generalized function on G, and gave a formula for its index. By analogy with the geometric quantization of symplectic manifolds, the virtual G-representation $Q(M) = [\ker \not{D}_b] - [\ker \not{D}_b^*]$ can be interpreted as the "quantization" of the contact manifold (M, E); the character of this representation is then given by the equivariant index of $\not{D}_b$. By defining contact analogues of the algebra of observables, prequantum line bundle and polarization, we further extend the analogy by giving a contact version of the Kostant–Souriau approach to quantization, and discussing the extent to which this approach is reproduced by the index-theoretic method.

1. Introduction

The problem of geometric quantization is well known in symplectic geometry, and dates back to the work of Souriau [1] and Kostant [2]. Symplectic geometry is the natural setting for classical Hamiltonian dynamics, but contact structures appear in classical physics as well: the role of contact geometry in Lagrangian mechanics is explained in [3], and the geometry of classical thermodynamics has a natural contact structure (see for example [4] or [5]). Quantization in contact geometry has also been studied, but usually, the methods employed are related to Berezin–Toeplitz quantization [6–8] rather than traditional geometric quantization. A brief sketch of an approach to geometric contact quantization was given by Vaisman in [9]; the first quantization we present for contact manifolds expands upon this sketch. We should also note that a geometric quantization for Jacobi manifolds has been given in [10] which specializes to contact manifolds. However, this approach is based on Vaisman's method of contravariant derivatives in Poisson geometry [11], while we make use of covariant derivatives, as is the norm in symplectic geometry.

In this article, we will describe two ways to define a "geometric quantization" of contact manifolds analogous to familiar methods in symplectic geometry. We first describe contact versions of the algebra of observables and Hamiltonian group actions, and give a construction of a Hilbert space of sections of a "quantum bundle" in the tradition of Kirillov–Kostant quantization. Tools from CR geometry play a significant role in this construction; in particular, this approach applies to Sasakian manifolds.

* Tel.: +1 510 642 6948; fax: +1 510 642 8204.
E-mail addresses: sean@math.berkeley.edu, dsfitzpatrick@gmail.com.
URL: http://math.berkeley.edu/~sean.

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The second approach is analogous to the use of Spin^c (almost complex) quantization in symplectic geometry as a model for geometric quantization in the Kähler case [12,13]: using a "compatible" almost CR structure, we construct an odd first order differential operator $\not{D}_b$ that reduces, in the case of a strongly pseudoconvex CR manifold, to the operator $\not{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*)$, where $\overline{\partial}_b$ is the tangential Cauchy–Riemann operator determined by the CR structure. The operator $\not{D}_b$ is not elliptic, but if a Lie group G acts on M transverse to the contact distribution, then $\not{D}_b$ will be transversally elliptic, and we can give a formula for its index similar to the Riemann–Roch formula in the symplectic case.

Our results can be summarized as follows: Let (M, E) be a compact cooriented contact manifold. A choice of contact form is given by a non-vanishing section θ of the annihilator line bundle $E^0 \subset T^*M$. (By assumption, E^0 is oriented, and hence, trivial.) The subbundle $E = \ker\theta \subset TM$ is a contact distribution if and only if $\mu_\theta = \theta \wedge d\theta^n/n!$ defines a volume form on M. If a compact Lie group G acts on M preserving E, the contact form θ can be assumed to be G-invariant by averaging, allowing us to define the contact momentum map $\Phi_\theta : M \to \mathfrak{g}^*$ given by

$$\langle \Phi_\theta, X\rangle = \theta(X_M)$$

for all $X \in \mathfrak{g}$, where X_M is the vector field generated by the infinitesimal action of X on M. The contact form also determines a Jacobi structure on M as follows: any vector field on M is determined uniquely by its pairings with θ and $d\theta$; in particular, the Reeb vector field ξ is defined by $\theta(\xi) = 1$ and $\iota(\xi)d\theta = 0$. This allows us to define a map $\Lambda^\# : T^*M \to E \subset TM$ by declaring that, for any $\eta \in T^*M$, we have

$$\theta(\Lambda^\#\eta) = 0 \quad \text{and} \quad \iota(\Lambda^\#\eta)d\theta = \eta(\xi)\theta - \eta.$$

Each $f \in C^\infty(M)$ is then associated to the Hamiltonian vector field $X_f = \Lambda^\# df + f\xi$, and the Jacobi bracket on $C^\infty(M)$ is given by $\{f, g\} = X_f \cdot g - g\xi \cdot f$. For any $f \in C^\infty(M)$, the associated Hamiltonian vector field satisfies $\mathcal{L}(X_f)\theta = (\xi \cdot f)\theta$, so that X_f is a contact vector field (see [14]). We then see that any group action on M preserving the contact form θ is Hamiltonian, in the following sense:

Theorem 1.1. *Suppose a compact Lie group G acts on a compact contact manifold M preserving the contact form θ. With respect to the Jacobi structure determined by θ, we have:*

1. *The map $\mathfrak{g} \to C^\infty(M)$ given by $X \mapsto \Phi_\theta^X$ is a Lie algebra homomorphism.*
2. *The Hamiltonian vector field associated to the function Φ_θ^X is equal to X_M.*

Thus, the diagram of Lie algebra homomorphisms

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & C^\infty(M) \\ & \searrow & \downarrow \\ & & \mathfrak{X}_{ham}(M) \end{array} \qquad (1)$$

commutes, where $\mathfrak{X}_{ham}(M)$ denotes the space of contact Hamiltonian vector fields on M. From [14], we know that the space of contact Hamiltonian vector fields is precisely the set of infinitesimal symmetries of the contact structure, as noted above: $\mathcal{L}(X_f)\theta = (\xi \cdot f)\theta$ for any $f \in C^\infty(M)$. We note that whenever $\xi \cdot f = 0$, X_f preserves the contact form, and hence the volume form μ_θ. These infinitesimal symmetries of the contact *form* are known as *quantomorphisms* in the case that M is a prequantum circle bundle [15]; in this case the contact form θ is a connection 1-form, and the functions $f \in C^\infty(M)$ such that $\xi \cdot f = 0$ can be identified with the pullback of functions on the base manifold M/S^1. In particular, since θ is preserved by the G-action, the momentum map components $\Phi_\theta^X = \langle \Phi_\theta, X\rangle \in C^\infty(M)$ satisfy $\xi \cdot \Phi_\theta^X = 0$ for all $X \in \mathfrak{g}$.

Proposition 1.2. *The space $C_b^\infty(M) = \{f \in C^\infty(M) | \xi \cdot f = 0\}$ is a Lie subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$, and the Jacobi bracket on $C^\infty(M)$ restricts to a Poisson bracket on $C_b^\infty(M)$.*

It is clear that the elements of $C_b^\infty(M)$ generate infinitesimal quantomorphisms (we will allow an abuse of language, and continue to apply this term to contact manifolds that are not prequantum circle bundles). Once an invariant contact form is chosen, we see that the momentum map components span a Lie subalgebra of $C_b^\infty(M) \subset C^\infty(M)$, and that the map (1) factors through $C_b^\infty(M)$.

To define a quantization of the contact manifold (M, θ), we used an adapted version of the *quantum bundles* from [16]. Since θ is a contact form, the 2-form $\Omega = -d\theta$ restricts to a symplectic structure on the subbundle $E = \ker\theta$. A Hermitian line bundle with connection $\pi : (\mathbb{L}, h, \nabla) \to (M, E, \Omega)$ will be called a quantum bundle if the curvature form of ∇ is equal to $i\Omega$. (In [16], it is only required that $\mathrm{curv}(\nabla) = i\Omega$ along the subbundle $E \subset TM$.) We can then construct the Hilbert space $\mathcal{H} = \Gamma_{L^2}(M, \mathbb{L})$ given by the L^2 completion of the space of smooth sections of $\mathbb{L}$ with respect to the inner product

$$\langle s_1, s_2\rangle = \int_M h(s_1, s_2)\mu_\theta.$$

It is then straightforward to check that the assignment

$$f \mapsto \nabla_{X_f} + i\pi^* f$$