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Apparent singularities of Fuchsian equations and the Painlevé property for Garnier systems

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ABSTRACT

We study movable singularities of Garnier systems using the connection of the latter with Schlesinger isomonodromic deformations of Fuchsian systems. Questions on the existence of solutions for some inverse monodromy problems are also considered.

1. Introduction

In the middle of the XIXth century, B. Riemann considered the problem of the construction of a linear differential equation

$$\frac{d^p u}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \cdots + b_p(z)u = 0 \qquad (1)$$

with the prescribed regular singularities $a_1, \ldots, a_n \in \overline{\mathbb{C}}$ (which are the poles of the coefficients) and prescribed monodromy.

Recall that a singular point a_i of Eq. (1) is said to be *regular* if any solution of the equation is of no more than a polynomial (with respect to $1/|z-a_i|$) growth in any sectorial neighbourhood of the point a_i.

By Fuchs's theorem [1] (see also [2, Th. 12.1]), a singular point a_i is regular if and only if the coefficient $b_j(z)$ has at this point a pole of order j or lower ($j = 1, \ldots, p$). Linear differential equations with regular singular points only are called *Fuchsian*.

Poincaré [3] has established that the number of parameters determining a Fuchsian equation of order p with n singular points is less than the dimension of the space $\mathcal{M}$ of monodromy representations, if $p > 2, n > 2$ or $p = 2, n > 3$ (see also [4, pp. 158–159]). Hence in the construction of a Fuchsian equation with the given monodromy there arise (besides $a_1, \ldots, a_n$) the so-called *apparent* singularities at which the coefficients of the equation have poles but the solutions are single-valued meromorphic functions, i. e., the monodromy matrices at these points are identity matrices. Below by apparent

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singular points of an equation we mean these very singularities. Thus, in general case the Riemann problem has a negative solution.

A similar problem for systems of linear differential equations is called the *Riemann–Hilbert problem*. This is the problem of the construction of a *Fuchsian system*

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \left(\sum_{i=1}^{n} \frac{B_i}{z-a_i}\right) y, \quad y(z) \in \mathbb{C}^p,\ B_i \in \mathrm{Mat}(p, \mathbb{C}), \tag{2}$$

with the given singularities $a_1, \ldots, a_n$ (if ∞ is not a singular point of the system, then $\sum_{i=1}^{n} B_i = 0$) and monodromy

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}, z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C}). \tag{3}$$

A counterexample to the Riemann–Hilbert problem was obtained by Bolibrukh (see [4, Ch. 5]). The solution of this problem has a more complicated history than that of the Riemann problem for scalar Fuchsian equations (before Bolibrukh it had long been wrongly regarded as solved in the affirmative; for details see [5]).

Alongside Fuchsian equations consider the famous non-linear differential equations—the *Painlevé* VI *equation* ($\mathrm{P_{VI}}$) and *Garnier systems*.

The equation $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$ is the non-linear differential equation

$$\begin{aligned}\frac{\mathrm{d}^2u}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{\mathrm{d}u}{\mathrm{d}t} \\ &\quad + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right)\end{aligned} \tag{4}$$

of second order with respect to the unknown function $u(t)$, where $\alpha, \beta, \gamma, \delta$ are complex parameters. This equation has three fixed singular points, $0, 1, \infty$. Its movable singularities (which depend on the initial conditions) can be poles only. In such a case one says that an equation satisfies the *Painlevé property*. The general $\mathrm{P_{VI}}$ equation (4) was first written down by Fuchs [6] and was added to the list of the equations now known as the *Painlevé* I–VI *equations* by Painlevé's student Gambier [7]. Among the non-linear differential equations of second order satisfying the Painlevé property, only the equations of this list in general case cannot be reduced to the known differential equations for elementary and classical special functions. The $\mathrm{P_{VI}}$ equation is the most general because all the other $\mathrm{P_{I-V}}$ equations can be derived from it by certain limit processes after the substitution of the independent variable t and parameters (see [8]).

The Garnier system $\mathcal{G}_n(\theta)$ depending on $n+3$ complex parameters $\theta_1, \ldots, \theta_{n+2}, \theta_\infty$ is a completely integrable system of non-linear partial differential equations of second order obtained by Garnier [9]. It was written down by Okamoto [8] in an equivalent Hamiltonian form

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \qquad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \quad i, j = 1, \ldots, n, \tag{5}$$

with certain Hamiltonians $H_i = H_i(a, u, v, \theta)$ rationally depending on $a = (a_1, \ldots, a_n), u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n), \theta = (\theta_1, \ldots, \theta_{n+2}, \theta_\infty)$. In the case $n = 1$ the Garnier system $\mathcal{G}_1(\theta_1, \theta_2, \theta_3, \theta_\infty)$ is an equivalent (Hamiltonian) form of $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$, where

$$\alpha = \frac{1}{2}\theta_\infty^2, \qquad \beta = -\frac{1}{2}\theta_2^2, \qquad \gamma = \frac{1}{2}\theta_3^2, \qquad \delta = \frac{1}{2}(1-\theta_1^2).$$

There exist classical results [6,9] on the connection of scalar Fuchsian equations of second order with $\mathrm{P_{VI}}$ equations and Garnier systems. Let us consider a scalar Fuchsian equation of second order with singular points $a_1, \ldots, a_n, a_{n+1} = 0, a_{n+2} = 1, a_{n+3} = \infty$ and apparent singularities $u_1, \ldots, u_n$ whose Riemann scheme has the form

$$\begin{pmatrix} a_i & \infty & u_k \\ 0 & \alpha & 0 \\ \theta_i & \alpha+\theta_\infty & 2 \end{pmatrix}, \quad i = 1, \ldots, n+2,\ k = 1, \ldots, n,\ \theta_i \notin \mathbb{Z}$$

(α depends on the parameters θ_i according to the classical Fuchs relation $\sum_{i=1}^{n+2} \theta_i + \theta_\infty + 2\alpha + 2n = 2n+1$). There is freedom of choice of such an equation. Its coefficients $b_1(z), b_2(z)$ depend on a, u, θ and n arbitrary parameters $v_1, \ldots, v_n$ ($v_i = \mathrm{res}_{u_i} b_2(z)$).

Fix a set θ ($\theta_i \notin \mathbb{Z}$) and consider an ($n$-dimensional) integral manifold M of the system $\mathcal{G}_n(\theta)$. Due to Okamoto's theorem [8], *Fuchsian equations corresponding to points $(a, u, v) \in M$ have the same monodromy.*[1] *Inversely, points (a, u, v) corresponding to Fuchsian equations with the same monodromy lie on the integral manifold of the system $\mathcal{G}_n(\theta)$.*

[1] This property is defined precisely in Section 3.