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String modular phases in Calabi–Yau families

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Dedicated to the memory of Max Kreuzer

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ABSTRACT

We investigate the structure of singular Calabi–Yau varieties in moduli spaces that contain a Brieskorn–Pham point. Our main tool is a construction of families of deformed motives over the parameter space. We analyze these motives for general fibers and explicitly compute the *L*-series for singular fibers for several families. We find that the resulting motivic *L*-functions agree with the *L*-series of modular forms whose weight depends both on the rank of the motive and the degree of the degeneration of the variety. Surprisingly, these motivic *L*-functions are identical in several cases to *L*-series derived from weighted Fermat hypersurfaces. This shows that singular Calabi–Yau spaces of non-conifold type can admit a string worldsheet interpretation, much like rational theories, and that the corresponding irrational conformal field theories inherit information from the Gepner conformal field theory of the weighted Fermat fiber of the family. These results suggest that phase transitions via non-conifold configurations are physically plausible. In the case of severe degenerations we find a dimensional transmutation of the motives. This suggests further that singular configurations with non-conifold singularities may facilitate transitions between Calabi–Yau varieties of different dimensions.

1. Introduction and outline

Previous work has provided support for the idea that the problem of deriving the geometry of spacetime in string theory from first principles can be approached via the concept of automorphic motives. The strategy of this framework is to relate automorphic forms derived from the geometry of spacetime to automorphic forms determined by the two-dimensional conformal field theory on the string worldsheet. It is possible to invert this process, and to construct the geometry of spacetime from the string theoretic automorphic forms. More precisely, it is possible to derive motivic pieces of the compactification variety from forms determined by the worldsheet theory. The combination of the resulting motivic building blocks then determines the global structure of the Calabi–Yau space. This approach thus leads to an explicit and computable realization of the concept of an emergent spacetime in string theory. The strategy outlined above has been pursued in the recent past in the context of weighted Fermat hypersurfaces and Gepner models in a number of papers (see e.g. [1–3] and references therein).

Weighted Fermat hypersurfaces, or Brieskorn–Pham varieties, are special in that the underlying conformal field theory is rational. Rationality of the theory implies that the algebra and its representations are under precise control and it is possible to understand the relationship between the worldsheet theory and the geometry of Calabi–Yau manifolds in some detail. In the early years after the original construction of Gepner [4] the techniques applied to this end came from Landau–Ginzburg theories [5–7] and non-linear sigma models [8]. The arithmetic-geometric approach pursued in [1,2] in the context of

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weighted Fermat hypersurfaces and Gepner models relates the geometric modular forms derived from motives of the Calabi–Yau varieties to Hecke indefinite modular forms. These in turn are built from the string functions that appear in the partition function of the exactly solvable model. This method thereby establishes a close connection between modular forms of rather different origins.

Since the constructions of Gepner and of Kazama and Suzuki two decades ago [4,9], it has proven difficult to extend the detailed understanding obtained for rational theories to deformations of such theories. Much less, therefore, is known about the conformal field theories that correspond to families of Calabi–Yau varieties. Our purpose in this paper is to initiate a program to address this problem by extending the modularity methods used for K3 surfaces and Calabi–Yau 3-folds of Brieskorn–Pham type in [1,2] to the class of Calabi–Yau families which contain a Brieskorn–Pham fiber in their moduli space. The structure of such families is of the form

$$X_n^d(\psi_D) = \left\{ (z_0 : \cdots : z_{n+1}) \in \mathbb{P}_{(w_0,\dots,w_{n+1})} \mid P(z_i, \psi_D) = p_{\rm BP}(z_i) - \sum_{J\in D} \psi_J z^J = 0 \right\} \tag{1}$$

where

$$p_{\rm BP}(z_i) = \sum_{i=0}^{n+1} z_i^{d_i} \tag{2}$$

is the weighted Fermat polynomial of degree d, with integral $d_i = d/w_i$. The monomials $z^J = \prod_{i=0}^{n+1} z_i^{j_i}$, with $J = (j_0, \dots, j_{n+1})$ for $j_i \in \mathbb{N}$ such that $\sum_{i=0}^{n+1} j_i w_i = d$, define the deformations. We denote the set of all deformation vectors J that appear in the polynomial by $D \subset \mathbb{Z}^{n+2}$.

In the context of this framework we analyze two separate but ultimately related issues. First, we generalize our previous results concerning the string theoretic interpretation of geometric modular forms from weighted Fermat hypersurfaces to families of Calabi–Yau varieties. Second, we use these generalized modularity results to investigate the conformal field theoretic structure of singular Calabi–Yau spaces.

The first step in this program involves the computation of geometric L-series defined via motives $M_\Omega(X(\psi_D))$ constructed from the families $X(\psi_D)$ defined by (1). Our construction of the motives of general families of n-dimensional weighted projective hypersurfaces involves the deformation of the Ω-motive defined at the Brieskorn–Pham point by the deformation monomials z^J, $J \in D$. For any vector J we define an action $r(J)$ of J on the Ω-motive $M_\Omega(X(0))$ of the Brieskorn–Pham variety $X(0)$, which we denote here by $r(J)M_\Omega(X(0))$. The deformed motive of the family determined by D is then defined as the sum of all contributions spanned by the action of the deformation vectors on the Ω-motive of the Brieskorn–Pham fiber

$$M_\Omega(X(\psi_D)) = \bigoplus_{J\in D} r(J) M_\Omega(X(0)). \tag{3}$$

This construction will be made precise in Section 3.

For smooth fibers the motives $M_\Omega(X(\psi_D))$ defined by (3) have in general higher rank and therefore are not modular in terms of congruence subgroups of the modular group $\mathrm{SL}(2,\mathbb{Z})$. The Langlands program suggests that such higher rank motives instead lead to automorphic forms associated with higher rank groups. We will see later in this paper that for singular fibers of the families of weighted projective hypersurfaces the motives (3) show interesting degeneration behavior. As a result, families of motives that are of high rank in the generic smooth fibers can become modular for singular Calabi–Yau fibers. This phenomenon can be used to investigate the modular structure of such singular configurations. In the past, most of the work on singular string compactifications has focused on conifolds, i.e. varieties with nodes. In particular, the D-brane analysis by Strominger, Greene and Morrison [10,11] of the geometric transitions introduced in [12], and analyzed in detail in [13–15], led to a paradigmatic interpretation.

A conifold singularity is not the only type of degeneration that can occur in moduli space, and the singularity types that we consider in the present paper are of more severe type, characterized by Milnor numbers that are larger than 1. For such more general singularities a D-brane interpretation is lacking, which raises the question whether degenerate Calabi–Yau varieties of non-conifold type admit a physical interpretation. For Brieskorn–Pham varieties the underlying worldsheet theory is a rational conformal field theory, and previous work has shown that the modular forms derived for such spaces have a string theoretic interpretation. The generalization of this string modularity program away from rational theories has not been addressed previously. The second purpose of this paper is to investigate the conformal field theoretic properties of the singular fibers in families of Calabi–Yau varieties. This leads to a surprising result for the deformed motives. As mentioned above, for general fibers of the Calabi–Yau families the deformed motive is of high rank, and therefore not modular. The unexpected phenomenon encountered in this work is that it can happen that the motivic L-series of a non-Fermat fiber in a family agrees with the motivic L-series of a weighted Fermat variety. We will call such pairs of models L-correlated. The existence of such L-correlated models shows that the deformed fiber inherits information from the rational conformal field theory underlying the Brieskorn–Pham manifold. This result makes it plausible that singular configurations of higher Milnor number could serve as links, much like conifolds, between different moduli spaces of smooth configurations.

A second phenomenon we encounter in the present investigation is that of a dimensional reduction of motives at singular fibers of families. It happens that in some families of varieties the modular forms of the deformed motives change their